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Some Shift Inequalities for Gaussian Measures

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ABSTRACT. Let μ be a centered Gaussian measure on a Banach space B and suppose $h \in H_\mu$, the generating Hilbert space of μ. If E is a Borel subset of B, we establish some inequalities between $\mu(E)$ and $\mu(E+h)$ which are similar in spirit to the isoperimetric inequality for Gaussian measures. We also include some applications to precise large deviation probabilities for μ.

1. Introduction

The Cameron-Martin formula provides a precise equality for shifts of a Gaussian measure, and here we present a related inequality. If γ_n is the canonical Gaussian measure on $\mathbb{R}^n$ the shift inequality takes the following form. Throughout $\Phi(\cdot)$ denotes the standard normal distribution on $\mathbb{R}^1$ and $\|\cdot\|_2$ the usual Euclidean norm on $\mathbb{R}^n$.

Theorem 1. Let γ_n be the canonical Gaussian measure on $\mathbb{R}^n$ and assume E is a Borel subset of $\mathbb{R}^n$. If $\theta \in (-\infty, \infty)$ is such that $\mu(E) = \Phi(\theta)$, then for every $\lambda \in \mathbb{R}^1$ and $h \in \mathbb{R}^n$ with $\|h\|_2 = 1$ we have

$$\Phi(\theta - |\lambda|) \leq \mu(E + \lambda h) \leq \Phi(\theta + |\lambda|). \tag{1.1}$$

Remark. If $\gamma_n(E) = 0$ (or $\gamma_n(E) = 1$), then by taking $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$ we see (1.1) is valid for all $\lambda \in \mathbb{R}^1$. Hence we assume $0 < \mu(E) < 1$ throughout. It is also easy to see from the proof that both inequalities in (1.1) can only be achieved by a half-space perpendicular to h, one for the upper bound and one for the lower bound, but as $|\lambda| \to \infty$ at least one side of the inequality always becomes trivial. Of course, the parameter λ in (1.1) can always be absorbed into the vector h without loss of generality, see (1.3) below.

By a monotone class argument, and some well-known properties of Gaussian measures, one can easily extend Theorem 1 to a Banach space. Hence we will not include details of the proof of this extension, but restrict ourself to a precise statement. Here B denotes a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*, and X is a centered B-valued Gaussian random vector with $\mu = \mathcal{L}(X)$. Hence μ is generated by a Hilbert space H_μ which is the closure of $S(B) = \left\{\int_B x f(x) d\mu(x) : f \in B^*\right\}$ in the inner product norm given on $S(B^*)$ by

$$\langle Sf, Sg \rangle_\mu = \int_B f(x) g(x) d\mu(x). \tag{1.2}$$

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We use $\|\cdot\|_\mu$ to denote the inner product norm induced on H_μ, and for well known properties, and various relationships between μ, H_μ, and B, consider Lemma 2.1 in [K]. These properties are used freely throughout the paper, as well as the fact that the support of μ is $\bar{H}_\mu$; the B-closure of H_μ. Then Theorem 1 implies:

Theorem 1′ Let μ be a centered Gaussian measure on B and assume E is a Borel subset of B. If $\theta \in (-\infty, \infty)$ is such that $\mu(E) = \Phi(\theta)$, then for every $h \in H_\mu$ we have

$$\Phi(\theta - \|h\|_\mu) \leq \mu(E+h) \leq \Phi(\theta + \|h\|_\mu). \tag{1.3}$$

Although we have never seen Theorem 1 or 1' in print, they are perhaps known by some experts. We learned this after circulating our Gaussian symmetrization proof of the result, and eventually several much simpler proofs emerged in discussions with Michel Ledoux. We thank him for his interest and contributions to these results. We present the simplest of these proofs below, and also some applications of the shift inequality in hopes that it will become more widely known. Our first application deals with the relationship of large deviation results for a Gaussian measure and the shift inequality. This gives another perspective to [KL]. The other applications are intuitive and easily believed, but we do not know how to prove them without the shift inequality. Also, our Theorem 2 in Section 3 provides a sharper result than Theorem 1 when the set E is convex or bounded. All these results are of isoperimetric type over different classes of sets.

2. Proof of Theorem 1

As mentioned earlier, our first proof used the Gaussian symmetrization of sets, but one can also prove the result using Ehrhard's symmetrization of functions developed in [E]. The proof we give here is based on the Cameron-Martin translation theorem as used in the proof of Theorem 3 by [KS].

Let $\langle x, y\rangle$ denote the canonical inner product on $\mathbb{R}^n$ and take $F = \{x \in \mathbb{R}^n : \langle x, h\rangle \leq \theta\}$ where $\gamma_n(F) = \Phi(\theta) = \gamma_n(E)$. Then by the Cameron-Martin theorem

$$\begin{aligned}\gamma_n(E+\lambda h)e^{\lambda^2\|h\|_2^2/2} &= \int_E e^{-\lambda\langle x,h\rangle}\, d\gamma_n(x) \\ &\leq \int_{E\cap F} e^{-\lambda\langle x,h\rangle}\, d\gamma_n(x) + \int_{E\cap F^c} e^{-|\lambda|\theta}\, d\gamma_n(x),\end{aligned} \tag{2.1}$$

and, similarly,

$$\gamma_n(F+\lambda h)e^{\lambda^2\|h\|_2^2/2} \geq \int_{E\cap F} e^{-\lambda\langle x,h\rangle}\, d\gamma_n(x) + \int_{E^c\cap F} e^{-|\lambda|\theta}\, d\gamma_n(x) \tag{2.2}$$

Since $\gamma_n(E) = \gamma_n(F)$, $\gamma_n(E\cap F^c) = \gamma_n(E) - \gamma_n(E\cap F)$, and $\gamma_n(E^c \cap F) = \gamma_n(F) - \gamma_n(E\cap F)$, we have

$$\gamma_n(E\cap F^c) = \gamma_n(E^c\cap F). \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) we see

$$\gamma_n(E + \lambda h) \leq \gamma_n(F + \lambda h) = \Phi(\theta + |\lambda|). \tag{2.4}$$

Thus the upper bound in (1.1) holds, and (1.1) follows from the following lemma.

Lemma. If the upper bound in (1.1) holds for all Borel subsets E and all $\lambda > 0$, then (1.1) holds as stated.

Proof. As mentioned previously, if $\mu(E) = 0$ (or $\mu(E) = 1$) the result holds for all $\lambda \in \mathbb{R}$ by setting $\theta = -\infty$ (or $\theta = +\infty$), so we assume $0 < \mu(E) < 1$. For any Borel set A with $0 < \mu(A) < 1$, we let $\theta(A) \in \mathbb{R}^1$ denote the unique number satisfying $\Phi(\theta(A)) = \mu(A)$.

Now $\mu(E + \lambda h) = 1 - \mu(E^c + \lambda h)$, and if $\lambda > 0$ the upper bound in (1.1) implies

$$\mu(E^c + \lambda h) \leq \Phi(\theta(E^c) + \lambda).$$

Now by symmetry we also have $\Phi(\theta(E^c) + \lambda) = 1 - \Phi(\theta(E) - |\lambda|)$ for $\lambda > 0$, so by combining the above when $\lambda > 0$ we have

$$\mu(E + \lambda h) \geq \Phi(\theta(E) - |\lambda|).$$

Thus the lower bound also holds for all $\lambda > 0$.

Now if both the upper bound and lower bound in (1.1) hold for all Borel sets E and all $\lambda > 0$, then it also holds for $\lambda < 0$. This follows since $\lambda < 0$ and μ symmetric implies

$$\mu(E + \lambda h) = \mu(E - |\lambda| h) = \mu(-E + |\lambda| h).$$

Symmetry also implies

$$\mu(-E) = \mu(E) = \Phi(\theta).$$

Hence (1.1) for $\lambda > 0$ implies

$$\Phi(\theta - |\lambda|) \leq \mu(-E + |\lambda| h) \leq \Phi(\theta + |\lambda|),$$

and combining the above we thus have for $\lambda < 0$ that

$$\Phi(\theta - |\lambda|) \leq \mu(E + \lambda h) \leq \Phi(\theta + |\lambda|).$$

Thus (1.1) holds for $\lambda < 0$ as well, and the lemma has been proved. Hence Theorem 1 is proven. $\square$

3. Additional Comments and Comparisons

Let A denote a symmetric (not necessarily convex) subset of $\mathbb{R}^n, h \in R^n$, and $S = \{x : |\langle x, h\rangle| \leq a\}$. If γ_n is the canonical Gaussian measure on $\mathbb{R}^n$ and $a \geq 0$ is such that $\gamma_n(A) = \gamma_n(S)$, then Theorem 3 of [KS] implies

$$\gamma_n(S + h) \leq \gamma_n(A + h). \tag{3.1}$$

Of course, this result also extends to their Theorem 3 using standard arguments just as Theorem 1 implies Theorem 1′. Combining (3.1) and the ideas in the proof of the previous lemma, we see

$$\gamma_n(S + h) \leq \gamma_n(A + h) \leq \gamma_n(T + h) \tag{3.2}$$

where $T = \{x : |\langle x, h\rangle| \geq b\}$ and $b \geq 0$ is such that $\gamma_n(A) = \gamma_n(T)$. Our Theorem 1 is equivalent to the following:

Theorem 1″. If E is a Borel subset of $\mathbb{R}^n$, $h \in \mathbb{R}^n$, $H_- = \{x : \langle x, h\rangle \leq a\}$, and $H_+ = \{x : \langle x, h\rangle \geq b\}$ where a and b are such that $\gamma_n(E) = \gamma_n(H_-) = \gamma_n(H_+)$, then

$$\gamma_n(H_+ + h) \leq \gamma_n(E + h) \leq \gamma_n(H_- + h). \tag{3.3}$$

Our next result is a more general form of Theorem 1″. In particular, it provides sharper estimates than Theorem 1″ when the set E is convex or bounded.

Theorem 2. Let E be a Borel subset of $\mathbb{R}^n$, $h \in \mathbb{R}^n$, and suppose

$$E \subseteq \{x : a \leq \langle x, h\rangle \leq d\}. \tag{3.4}$$

If $S_- = \{x : a \leq \langle x, h\rangle \leq b\}, S_+ = \{x : c \leq \langle x, h\rangle \leq d\}$, are such that

$$\gamma_n(S_-) = \gamma_n(S_+) = \gamma_n(E),$$

then

$$\gamma_n(S_+ + h) \leq \gamma_n(E + h) \leq \gamma_n(S_- + h). \tag{3.5}$$

Proof. By the Cameron-Martin formula we have

$$\gamma_n(E + h) = e^{-\|h\|_2^2/2} \int_E e^{-\langle x,h\rangle} d\gamma_n(x) \tag{3.6}$$

and

$$\gamma_n(S_- + h) = e^{-\|h\|_2^2/2} \int_{S_-} e^{-\langle x,h\rangle} d\gamma_n(x). \tag{3.7}$$

Furthermore,

$$\int_E e^{-\langle x,h\rangle} d\gamma_n(x) = \int_{E\cap S_-} e^{-\langle x,h\rangle} d\gamma_n(x) + \int_{E\cap S_-^c} e^{-\langle x,h\rangle} d\gamma_n(x), \tag{3.8}$$

$$\int_{S_-} e^{-\langle x,h\rangle} d\gamma_n(x) = \int_{E\cap S_-} e^{-\langle x,h\rangle} d\gamma_n(x) + \int_{E^c\cap S_-} e^{-\langle x,h\rangle} d\gamma_n(x), \tag{3.9}$$

and for all $x \in E \cap (S_-)^c, y \in S_- \cap E^c$ we have $\langle x,h\rangle \geq b \geq \langle y,h\rangle \geq a$. Hence

$$\begin{aligned}\int_{E\cap S_-^c} e^{-\langle x,h\rangle} d\gamma_n(x) &\leq e^{-b}\gamma_n(E\cap S_-^c)\\ &= e^{-b}\gamma_n(S_-\cap E^c)\\ &\leq \int_{S_-\cap E^c} e^{-\langle y,h\rangle} d\gamma_n(y),\end{aligned} \tag{3.10}$$

since $\gamma_n(E\cap S_-^c) = \gamma_n(S_-\cap E^c)$. Combining (3.6)–(3.10) we thus have

$$\gamma_n(E+h) \leq \gamma_n(S_- + h),$$

which is the upper bound in (3.5). The proof of the lower bound is similar, and hence Theorem 2 is proven. □

As a consequence of Theorem 2, we give the following result which provides a sharper estimate than Anderson's inequality for the upper bound, and the well known result for the lower bound regarding the shift of symmetric convex set. Of course, Theorem 2 applies to arbitrary Borel (not necessarily symmetric) sets as well.

Corollary. Let C be a symmetric convex subset of $\mathbb{R}^n$ and $h \in \mathbb{R}^n$. Then

$$\max(\gamma_n(S_+ + h), \exp(-\|h\|_2^2/2)\gamma_n(C)) \leq \gamma_n(C+h) \leq \min(\gamma_n(S_- + h), \gamma_n(C)) \tag{3.11}$$

where

$$S_- = \left\{x : -\max_{y\in C}\langle y,h\rangle \leq \langle x,h\rangle \leq b\right\}, S_+ = \left\{x : c \leq \langle x,h\rangle \leq \max_{y\in C}\langle y,h\rangle\right\},$$

are such that $\gamma_n(S_-) = \gamma_n(S_+) = \gamma_n(C)$.

Proof. Since C is symmetric convex, we have

$$C \subseteq \left\{x : -\max_{y\in C}\langle y,h\rangle \leq \langle x,h\rangle \leq \max_{y\in C}\langle y,h\rangle\right\}.$$

Thus (3.11) follows from Theorem 2 and the following well known facts about the shift of symmetric convex set (see, for example, [DHS]):

$$\exp(-\|h\|_2^2/2)\gamma_n(C) \leq \gamma_n(C+h) \leq \gamma_n(C). \tag{3.12}$$

Note that if the symmetric convex set C is unbounded in the direction of h, then we can simply take $d = \infty$ in (3.4) and the set S_+ is a half space in this case.

It is easy to see that in a variety of cases our new bounds are better than the simple but very useful facts given in (3.12), in particular when $\|h\|_2$ is large. This is obvious in terms of the upper bound, but the lower bound is also better when $c < 0$ and $\|h\|_2$ is large. This is interesting since one also knows that

$$\gamma_n(C+h) \sim \exp(-\|h\|_2^2/2)\gamma_n(C) \text{ as } \|h\|_2 \to \infty.$$

As can be seen from Theorem 2, the bounds provided here take into account the relative size of C in the direction of the shift, as well as the magnitude of $\gamma_n(C)$ and $\|h\|_2$.

Now we examine how these results relate to "isoperimetric inequalities" over different classes of sets, and mention some related open problems from this point of view. Let $\mathcal{B}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{C}(\mathbb{R}^n)$ denote the class of Borel sets, symmetric Borel sets and convex sets in $\mathbb{R}^n$, respectively. Fix $0 \le \alpha \le 1$. Theorem 1 tells us

$$\sup\{\gamma_n(E+h) \,:\, \gamma_n(E) = \alpha, E \in \mathcal{B}(\mathbb{R}^n)\} = \gamma_n(H_- + h) \tag{3.13}$$

for any $h \in \mathbb{R}^n$ where $H_- = \{x : \langle x, h\rangle \le a\}$ is the half space such that $\gamma_n(H_-) = \alpha$, and

$$\inf\{\gamma_n(E+h) \,:\, \gamma_n(E) = \alpha, E \in \mathcal{B}(\mathbb{R}^n)\} = \gamma_n(H_+ + h) \tag{3.14}$$

for any $h \in \mathbb{R}^n$ where $H_+ = \{x : \langle x, h\rangle \ge b\}$ is the half space such that $\gamma_n(H_+) = \alpha$. The extremal set in (3.13) is given by H_- and by H_+ in (3.14). They are unique up to sets of measure zero.

Our Theorem 2 implies in particular that for any $h \in \mathbb{R}^n$

$$\begin{aligned}&\sup\{\gamma_n(C+h) \,:\, \gamma_n(C) = \alpha, C \in \mathcal{C}(\mathbb{R}^n) \text{ and } C \subseteq \{x : a \le \langle x, h\rangle \le d\}\} \\ &= \gamma_n(S_- + h)\end{aligned} \tag{3.15}$$

where $S_- = \{x : a \le \langle x, h\rangle \le b\}$ is the slab such that $\gamma_n(S_-) = \alpha$, and for any $h \in \mathbb{R}^n$

$$\begin{aligned}&\inf\{\gamma_n(C+h) \,:\, \gamma_n(C) = \alpha, C \in \mathcal{C}(\mathbb{R}^n) \text{ and } C \subseteq \{x : a \le \langle x, h\rangle \le d\}\} \\ &= \gamma_n(S_+ + h)\end{aligned} \tag{3.16}$$

where $S_+ = \{x : c \le \langle x, h\rangle \ge d\}$ is the slab such that $\gamma_n(S_+) = \alpha$. The extremal sets here are given by S_- in (3.15) and by S_+ in (3.16), and again are unique in the sense indicated above.

Equation (3.2) implies in particular that

$$\sup\{\gamma_n(A+h) \,:\, \gamma_n(A) = \alpha, A \in \mathcal{S}(\mathbb{R}^n)\} = \gamma_n(P_- + h) \tag{3.17}$$

for any $h \in \mathbb{R}^n$ where $P_- = \{x : |\langle x, h\rangle| \le a\}$ and a is such that $\gamma_n(P_-) = \alpha$, and

$$\inf\{\gamma_n(A+h) \,:\, \gamma_n(A) = \alpha, A \in \mathcal{S}(\mathbb{R}^n)\} = \gamma_n(P_+ + h) \tag{3.18}$$

for any $h \in \mathbb{R}^n$ where $P_+ = \{x : |\langle x, h\rangle| \ge b\}$ and b is such that $\gamma_n(P_+) = \alpha$. The extremal sets are given by P_- in (3.17) and by P_+ in (3.18), and are unique as before.

Thus for the simple shift operation there are a variety of isoperimetric results over different classes of sets. On the other hand, there are other useful operations on sets where much less is known at present. Here we only mention two well known ones, namly, addition and dilation of sets.

The isoperimetric property for Gaussian measures states that

$$\inf\{\gamma_{n*}(E+\lambda K) : \gamma_n(E)=\alpha, E\in\mathcal{B}(\mathbb{R}^n)\} = \gamma_n(H+\lambda K) \tag{3.19}$$

for any $\lambda \geq 0$ where $K = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ and H is a half space such that $\gamma_n(H) = \alpha$. Here $A + \lambda K = \{a + \lambda k : a \in A, k \in K\}$, and $\gamma_{n*}(\cdot)$ is the inner measure obtained from γ_n. The relation (3.19) is very powerful, and provides the best results in a variety of settings. It is due independently to Borell[Bo] and Sudakov-Tsierlson[ST]. A beautiful extension for convex Borel sets using Gaussian symmetrizations was given by Ehrhard in [E]. If the inf in (3.19) is replaced by sup, then the righthand term easily is seen to be one. On the other hand, it seems to be a very hard problem (also for sup instead of inf) if we replace $E \in \mathcal{B}(\mathbb{R}^n)$ in (3.19) by $E \in \mathcal{S}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$. In particular, the extremal sets depend on the parameter α and the number $\lambda > 0$ as can be easily seen in $\mathbb{R}^2$. For example, let $K_b = \{(x,y) : x^2+y^2 \leq b^2\}$ and $S_a = \{(x,y) : |x| \leq a\}$ where a and b are such that $f(b) = \gamma_2(K_b) = \gamma_2(S_a) = g(a)$. Note that

$$f(b) = \frac{1}{2\pi}\int\int_{x^2+y^2\leq b^2} e^{-(x^2+y^2)/2}dxdy = 1 - e^{-b^2/2},$$

$$g(a) = 1 - \frac{2}{\sqrt{2\pi}}\int_0^a e^{-x^2/2}dx$$

and

$$\gamma_2(K_b+\lambda K) = \gamma_2(K_{b+\lambda}) = f(b+\lambda), \quad \gamma_2(S_a+\lambda K) = \gamma_2(S_{a+\lambda}) = g(a+\lambda).$$

It is easy to see that with fixed $b > a > 0$ from $f(b) = g(a)$, $f(b+\lambda) > g(a+\lambda)$ for λ sufficiently large. On the other hand, we have

$$\frac{\partial}{\partial\lambda}f(b+\lambda)\Big|_{\lambda=0} = be^{-b^2/2} = b(1-f(b)) = b(1-g(a)) = \frac{2b}{\sqrt{2\pi}}\int_a^\infty e^{-x^2/2}dx$$

$$< \frac{2}{\sqrt{2\pi}}e^{-a^2/2} = \frac{\partial}{\partial\lambda}g(a+\lambda)\Big|_{\lambda=o}$$

for $a > 0$ sufficiently small since $b > a$ is also very small. It should also be noted that the first inequality conjectured in Problem 3 of the book [Lif], page 277, is false. The extremal set is neither a slab nor a ball depending on different values of α and λ as seen in the above example.

For the dilation operation it is known that any fixed $0 \leq \alpha \leq 1$

$$\inf\{\gamma_{n*}(\lambda E) : \gamma_n(E)=\alpha, E\in\mathcal{B}(\mathbb{R}^n)\} = \gamma_n(\lambda H) \tag{3.20}$$

for any $\lambda \geq 1$ where H is a half space such that $\gamma_n(H) = \alpha$. The relation was first given in [LS] for $\gamma_n(E) \geq 1/2$ in connection to the exponential integrability of seminorms of Gaussian random vectors. On the other hand, it is still a conjecture if we replace $E \in \mathcal{B}(\mathbb{R}^n)$ in (3.20) by $E \in \mathcal{S}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$. It was shown in [KS] that if the set E is totally symmetric, that is, symmetric with respect to each coordinate, then the conjecture holds.

4. Some Applications of the Shift Theorem

Our first application involves the connection between large deviation probabilities and the shift inequality for Gaussian measures. If a_t and b_t are non-negative, we write $a_t << b_t$ if $\overline{\lim}_{t\to\infty} a_t/b_t < \infty$, and $a_t \approx b_t$ if both $a_t << b_t$ and $b_t << a_t$. Here we assume D to be an open convex subset of B and μ is a centered Gaussian measure on B. The parameter t is strictly positive. If $D \cap \bar{H}_\mu \neq$ and $0 \notin \bar{D}$, then Proposition 1 of [KL] showed there exists a unique point $h \in \partial D$ and $f \in B^*$ such that $h = Sf$, $D \subseteq \{x : f(x) > f(h)\}$, and $\inf_{x\in D} \|x\|_\mu^2 = \inf_{x\in \bar{D}} \|x\|_\mu^2 = \|h\|_\mu^2$. The point h is called a dominating point of D. Hence by the Cameron-Martin theorem

$$\mu(tD) = \mu(t(D-h)+th) = \exp\left\{-t^2\|h\|_\mu^2/2\right\} \int_{t(D-h)} e^{-tf(x)} d\mu(x) \tag{4.1}$$

where $h = Sf$. Since $f(x) > f(h)$ for all $x \in D$ and f is a centered Gaussian variable it is easy to see that

$$\mu(tD) << t^{-1} \exp\left\{-t^2\|h\|_\mu^2/2\right\}, \tag{4.2}$$

which provides an upper bound on $\mu(tD)$.

For a lower bound we consider the lower bound in the shift inequality applied to the middle term in (4.1). This implies

$$\mu(tD) \geq \Phi(\theta_t - t\|h\|_\mu) \tag{4.3}$$

where θ_t satisfies

$$\mu(t(D-h)) = \Phi(\theta_t). \tag{4.4}$$

Since $t(D-h)$ is a subset of $\{x : f(x) > 0\}$ for $t > 0$ we have $\mu(t(D-h)) \leq 1/2$ and thus $\theta_t \leq 0$. Hence if D is also an open ball in a 2-smooth Banach space, then Corollary 1 of [KL] implies

$$\overline{\lim_{t\to\infty}}\, t(\frac{1}{2} - \mu(t(D-h))) < \infty. \tag{4.5}$$

Now (4.4) and (4.5) with $\theta_t \leq 0$ combine to imply $0 \leq -\theta_t \leq t^{-1}$ as $t \to \infty$. Hence (4.2) and (4.3) imply as $t \to \infty$

$$\mu(tD) \approx t^{-1} \exp\{(\theta_t - t\|h\|_\mu)^2/2\} \approx t^{-1} \exp\{-t^2\|h\|_\mu^2/2\}.$$

Of course, the crucial things to prove in the above are that the dominating point exists, and (4.5) holds. This is done in [KL], but once these things are known, the shift inequality applies nicely for the lower bound. This is the more delicate part of the argument, and what was done in [KL] was to show the integral in (4.1) to be of size t^{-1} as $t \to \infty$. These approaches are essentially equivalent, but the shift inequality is definitely more direct.

If U is the open unit ball of B in the norm $\|\cdot\|$, then easy examples in $\mathbb{R}^n$ show that for various norms $\|\cdot\|$ we do not have

$$\lim_{t\to\infty} P(X \in t(U-p)) = 1/2 \tag{4.6}$$

for every $p \in \mathbb{R}^n$, $\|p\| = 1$. For example, if $\mathcal{L}(X) = \mu$, where μ is the standard normal distribution on $\mathbb{R}^n$, and $\|\cdot\|$ is an ℓ^∞ or ℓ^1 norm, then the limit in (4.6) is 2^{-d} when p is one of the corners of the closed unit ball $\overline{U}$. On the other hand, our next result will show that there always exists p, $\|p\| = 1$, such that (4.6) holds. Hence at such p the boundary of U is rather flat. This is obvious in many cases, and intuitively clear, but the result below is completely general. Also, we do not know how to prove this result without the shift inequality.

Proposition. Let U be the open unit ball of B. Then there exists $h \in \partial U$ such that

$$\lim_{t\to\infty} \mu(t(U-h)) = 1/2. \tag{4.7}$$

Proof. Let K be the unit ball of H_μ, $D = U^c$, and recall μ is a centered non-degenerate Gaussian measure. Hence $K \neq \{0\}$ and if

$$\lambda_0 = \sup\{\lambda > 0 : \lambda K \subseteq \overline{U}\},$$

we have $\lambda_0 K \cap D \neq \phi$. Hence take $h \in \lambda_0 K \cap D$. Then $\|h\|_\mu = \lambda_0 > 0$, $\|h\| = 1$, and by Lemma 2.1 in [K] we have

$$\sigma^2 = \sup_{\|f\|_{B^*}\le 1} \int_B f^2(x)d\mu(x) = \sup_{x\in K} \|x\|^2.$$

Thus $\sigma^2 = \lambda_0^{-2}$, and [LT, p. 87] implies that, as $t \to \infty$,

$$\mu(tD) = \exp\{-t^2\|h\|_\mu^2/2 + \epsilon(t)t\} \tag{4.8}$$

where $\lim_{t\to\infty} \epsilon(t) = 0$. Hence if $\mu(tD) = \Phi(\theta_t)$, then $\lim_{t\to\infty} \theta_t = -\infty$ and we also have

$$\mu(tD) = \exp\{-\theta_t^2/2 - \log\sqrt{2\pi} - \log|\theta_t| + \delta(t)\} \tag{4.9}$$

where $\lim_{t\to\infty} \delta(t) = 0$. Combining (4.8) and (4.9) it is easy to check that

$$\theta_t = -t\|h\|_\mu + \gamma(t) \tag{4.10}$$

where $\lim_{t\to\infty} \gamma(t) = 0$. Thus

$$\epsilon(t)t = t\|h\|_\mu\gamma(t) - \gamma^2(t)/2 - \log\sqrt{2\pi} - \log|-t\|h\|_\mu + \gamma(t)| + \delta(t),$$

and hence

$$\gamma(t) = \frac{\epsilon(t)}{\|h\|_\mu} + t^{-1}\log(\sqrt{2\pi}\|h\|_\mu t) + o(t^{-1}) \tag{4.11}$$

as $t \to \infty$. Now $\mu(t(U-h)) = 1 - \mu(t(D-h))$, and by the shift inequality and (4.10) we have

$$1/2 \le \mu(t(D-h)) \le \Phi(\theta_t + |t|\|h\|_\mu) = \Phi(\gamma(t)) \le 1/2 + \gamma(t). \tag{4.12}$$

Furthermore, (4.12) implies $\gamma(t)$ as given in (4.1) is non-negative as $t \to \infty$. Taking complements, as $t \to \infty$

$$1/2 - \gamma(t) \leq \mu(t(U - h)) \leq 1/2$$

where $\lim_{t\to\infty} \gamma(t) = 0$. Thus (4.7) holds. □

Finally we present two miscellaneous inequalities which are intuitively obvious, but seemingly not so easy to prove without the shift inequality. They are as follows.

I. If E is *any* norm bounded Borel subset of B with $\mu(E) \geq 1/2$ and $h \in H_\mu$, $\|h\|_\mu = 1$, then $\mu(E^c + \lambda h)$ converges to one as $\lambda \to \infty$ at a rate slower than the μ-measure of the half space $\{x \in B : \langle x, h\rangle^\sim \leq \lambda\}$. This follows immediately from Theorem 1 since for $\lambda > 0$

$$\mu(E^c + \lambda h) \leq \Phi(\theta + \lambda)$$

where $\Phi(\theta) = \mu(E^c) \leq 1/2$. Thus $\theta \leq 0$ and

$$\mu(E^c + \lambda h) \leq \Phi(\lambda) = \mu(x : \langle x, h\rangle^\sim \leq \lambda)$$

since $\langle x, h\rangle^\sim$ is $N(0,1)$ when $\|h\|_\mu = 1$.

II. Let C be an open cone strictly smaller than a half space with vertex at the zero vector in B and assume $\mu(C) > 0$. If $h \in H_\mu \cap C$, $\|h\|_\mu = 1$, it is easy to see that $\lim_{\lambda\to\infty} \mu(C + \lambda h) = 0$ and $\lim_{\lambda\to\infty} \mu(C + \lambda(-h)) = 1$. Since the map $\lambda \to \mu(C + \lambda h)$ is easily seen to be continuous by the Cameron-Martin formula and the dominated convergence theorem, we take λ_0 such that $\mu(C + \lambda_0 h) = 1/2$. Then, as $\lambda \to \infty$,

$$\begin{aligned}\mu((C + \lambda(-h))^c) &= 1 - \mu(C + \lambda(-h)) \\ &= 1 - \mu(C + \lambda_0 h + (\lambda + \lambda_0)(-h)) \\ &\geq 1 - \Phi(\lambda + \lambda_0)\end{aligned}$$

by the upper bound in the shift inequality for $E = C + \lambda_0 h$. Hence as $\lambda \to \infty$, $\mu((C + \lambda(-h))^c)$ goes to zero faster than the μ-measure of the half space $\{x : \langle x, h\rangle^\sim \leq -\lambda_0 - \lambda\}$.

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