Journal of Theoretical Probability, Vol. 6, No. 3, 1993

Small Ball Estimates for Brownian Motion and the Brownian Sheet

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Received April 14, 1992; revised October 12, 1992

Small ball estimates are obtained for Brownian motion and the Brownian sheet when balls are given by certain Hölder norms. As an application of these results we include a functional form of Chung's LIL in this setting.

KEY WORDS: Brownian motion; Brownian sheet; Gaussian samples; Hölder norm; Chung's functional LIL.

1. INTRODUCTION

Small ball estimates for Brownian motion and the Brownian sheet are obtained when the balls are given by certain Hölder norms. As an application, we establish some lim inf results for Gaussian samples of these processes, as well as Chung's LIL in this setting, but now we introduce some notation.

Let $I = [0, 1]$, with $C_0(I)$ the continuous real-valued function $f(\cdot)$ on I such that $f(0) = 0$. Then the usual sup-norm is given by

$$\|f\|_\infty = \sup_{0 \leqslant t \leqslant 1} |f(t)|$$

and the α-Hölder norm, $0 < \alpha \leqslant 1$, is

$$\|f\|_\alpha = \sup_{\substack{s, t \in I \\ s \neq t}} \frac{|f(t) - f(s)|}{|s - t|^\alpha} \tag{1.1}$$

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0894-9840/93/0700-0547$07.00/0

Of course, $\|f\|_\alpha$ is not finite for all $f \in C_0(I)$, so we define the spaces

$$H_\alpha(I) = \{f \in C_0(I) \colon \|f\|_\alpha < \infty\}$$

and

$$H_{\alpha,0}(I) = \left\{f \in C_0(I) \colon \lim_{\delta \to 0} \sup_{\substack{s,t \in I \\ 0 < |s-t| \leqslant \delta}} \frac{|f(t)-f(s)|}{|s-t|^\alpha} = 0\right\}$$

Since $f(0) = 0$ for all $f \in C_0(I)$, $\|f\|_\alpha$ is actually a norm, and both H_α and $H_{\alpha,0}$ are Banach spaces in $\|\cdot\|_\alpha$ with $\|f\|_\infty \leqslant \|f\|_\alpha$.

It is well known that sample continuous Brownian motion in R^1, which we denote by $\{W(t) \colon t \geqslant 0\}$, satisfies $P(\|W\|_\alpha < \infty) = 1$ for $0 < \alpha < 1/2$, and here we study the behavior of

$$P(\|W\|_\alpha \leqslant \varepsilon) \qquad \text{as} \qquad \varepsilon \to 0 \tag{1.2}$$

The asymptotics of Eq. (1.2) when $\|\cdot\|_\alpha$ is replaced by the sup-norm, or L_2-norm, are well known classical results. However, the study of Eq. (1.2) with $0 < \alpha < 1/2$ appears to be new.

We also examine an analogue of Eq. (1.2) for the Brownian sheet. The sample continuous Brownian sheet is the centered Gaussian process denoted by $\{W(s,t) \colon s, t \geqslant 0\}$ and satisfying $E(W^2(s,t)) = st$. If $I^2 = I \times I$ and $C_0(I^2)$ denotes the continuous real-valued functions $f(\cdot,\cdot)$ on I^2 such that $f(0,t) = f(s,0) = 0$ for $0 \leqslant s, t \leqslant 1$, then $\{W(s,t) \colon 0 \leqslant s, t \leqslant 1\}$ takes values in $C_0(I^2)$ with probability one. Furthermore, for $0 < \alpha < 1/2$ the norm

$$\|f\|_\alpha = \sup_{\substack{(s,t),(s',t') \in I^2 \\ (s,t) \neq (s',t')}} \frac{|f(s,t) - f(s',t')|}{((s-s')^2 + (t-t')^2)^{\alpha/2}} \tag{1.3}$$

again satisfies $P(\|W(\cdot,\cdot)\|_\alpha < \infty) = 1$.

Also, if the single differences in Eq. (1.3) are replaced by suitable double differences we obtain a still larger α-Hölder norm. This is the norm we study for the Brownian sheet. That is, if $(s,t) \in I^2$ and $(s+h, t+h') \in I^2$ with $h, h' > 0$ we set

$$\Delta f(s,t,h,h') = f(s+h, t+h') - f(s, t+h') - f(s+h, t) + f(s,t) \tag{1.4}$$

and for $0 < \alpha < 1/2$ we define the norm q_α by

$$q_\alpha(f) = \sup_{\substack{(s,t) \in I^2 \\ (s+h, t+h') \in I^2 \\ h, h' > 0}} \frac{|\Delta f(s,t,h,h')|}{(hh')^\alpha} \tag{1.5}$$

Defining the analogues of $H_\alpha(I)$ and $H_{\alpha,0}(I)$ for the norm $q_\alpha(\cdot)$ we set

$$H_{q_\alpha}(\cdot) = \{f \in C_0(I^2): q_\alpha(f) < \infty\}$$

and

$$H_{q_\alpha,0}(I^2) = \left\{ f \in C_0(I^2): \lim_{\delta \to 0} \sup_{\substack{(s,t) \in I^2, (s+h, t+h') \in I^2 \\ h, h' > 0, \max(h,h') \leqslant \delta}} \frac{|\Delta f(s,t,h,h')|}{(hh')^\alpha} = 0 \right\}$$

Then $H_{q_\alpha}(I^2)$ and $H_{q_\alpha,0}(I^2)$ are both Banach spaces in the norm $q_\alpha(\cdot)$, and Orey and Pruitt[13] [Theorem 2.1] implies

$$P(q_\alpha(W) < \infty) = 1 \tag{1.6}$$

for all $\alpha \in (0, 1/2)$. Furthermore, since $q_\alpha(f)$ increases as α increases, we also have paths of the Brownian sheet in $H_{q_\alpha,0}$ with probability one for each $\alpha \in (0, 1/2)$.

The small ball estimates we obtain are given in the following theorems. The main tool for Theorem 1.1 is a result of Ciesielski[6] which establishes Banach space isomorphisms between the spaces of α-Hölder paths and sequence spaces. This method has also been used recently in the study of some large deviation results for Brownian motion in Baldi and Roynette,[2] and we are indebted to Baldi and Roynette[2] for inspiring us to use this method in connection with these small ball problems and to Michel Ledoux for pointing this manuscript out. It is well known that small ball problems are very different than their large deviation counter parts, but nevertheless the Ciesielski isomorphisms are useful in both contexts.

In Theorem 1.2 we provide small ball estimates for $q_\alpha(W)$ when W is the Brownian sheet and $0 < \alpha < 1/2$. If the usual L_2-norm on I^2 is used, then small ball estimates are also known for the Brownian sheet, but for the sup-norm the problem remains open. Hence the results in Theorem 1.2 extend what we know for the Brownian sheet, but much remains to be done. The recent paper of Li[12] contains additional information and further references.

Theorem 1.1. Let $\{W(t): t \geqslant 0\}$ be a sample continuous Brownian motion in R^1 and set

$$\psi_\alpha(\varepsilon) = \log P(\|W\|_\alpha \leqslant \varepsilon) \qquad \varepsilon > 0 \tag{1.7}$$

If $0 < \alpha < 1/2$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(1-2\alpha)} \psi_\alpha(\varepsilon) = -C_\alpha \tag{1.8}$$

exists with

$$2^{-2(1-\alpha)/(1-2\alpha)}\Gamma_\alpha \leqslant C_\alpha \leqslant (2^{-1/2}(2^\alpha-1)(2^{1-\alpha}-1))^{-2/(1-2\alpha)}\Gamma_\alpha \tag{1.9}$$

where

$$\Gamma_\alpha = (2/\pi)^{1/2}\int_0^\infty \frac{u^{2/(1-2\alpha)}e^{-u^2/2}}{1-G(u)}\,du \quad \text{and} \quad G(u)=(2/\pi)^{1/2}\int_u^\infty e^{-x^2/2}\,dx \tag{1.10}$$

Remark 1.1. The precise value of C_α is unknown to us.

To state our result for the Brownian sheet we need the notation $f(\varepsilon)\approx g(\varepsilon)$ as $\varepsilon\to 0$, which means that there is a constant C, $1<C<\infty$, such that

$$1/C \leqslant \varliminf_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) \leqslant \varlimsup_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) \leqslant C \tag{1.11}$$

Similar notation is used for functions $f(x)$ as $x\to\infty$, and also used for sequences. We will write $f(\varepsilon)\sim g(\varepsilon)$ as $\varepsilon\to 0$ if $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon)=1$. Hence Eq. (1.8) can be written as

$$\psi_\alpha(\varepsilon) \sim -C_\alpha \varepsilon^{-2/(1-2\alpha)} \tag{1.12}$$

Theorem 1.2. Let $\{W(s,t): s,t\geqslant 0\}$ be a sample continuous Brownian sheet and set

$$\psi_{q_\alpha}(\varepsilon)=\log P(q_\alpha(W)\leqslant\varepsilon) \qquad \varepsilon>0 \tag{1.13}$$

If $0<\alpha<1/2$, then as $\varepsilon\to 0$

$$\psi_{q_\alpha}(\varepsilon)\approx -\varepsilon^{-2/(1-2\alpha)}\log(1/\varepsilon) \tag{1.14}$$

More precisely, for all $\varepsilon>0$ sufficiently small

$$\psi_{q_\alpha}(\varepsilon)\geqslant -(2^{-(1-\alpha)}(2^\alpha-1)(2^{1-\alpha}-1))^{-4/(1-2\alpha)}/(1-2\alpha)\cdot\Gamma_\alpha\varepsilon^{-2/(1-2\alpha)}\log(1/\varepsilon)$$

and

$$\psi_{q_\alpha}(\varepsilon)\leqslant -2^{4(1-\alpha)/(1-2\alpha)}/(1-2\alpha)\cdot\Gamma_\alpha\varepsilon^{-2/(1-2\alpha)}\log(1/\varepsilon)$$

Remark 1.2. After completing this manuscript we became aware of Baldi and Roynette,[3] which was kindly supplied to us by Michel Ledoux. The results in Baldi and Roynette[2] and Baldi and Roynette[3] have now been merged, and will appear as Baldi and Roynette.[4] The overlap between Baldi and Roynette[4] and this paper consists of Theorem 1.1 and Eq. (4.5) in Theorem 4.1.

2. PROOF OF THEOREM 1.1

Our first proposition establishes that the limit in Eq. (1.8) exists, and that it is a certain infimum. Unfortunately, we are unable to evaluate this infimum, so we turn to the task of giving the bounds for C_α immediately following the proposition.

Proposition 2.1. Let $\{W(t): t \geqslant 0\}$ be a sample continuous Brownian motion in R^1. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(1-2\alpha)} \log P(\|W(t)\|_\alpha \leqslant \varepsilon) = \inf_{x>0} x^{2/(1-2\alpha)} \log P(\|W(t)\|_\alpha \leqslant x) \tag{2.1}$$

Proof. First we show that for all $\varepsilon > 0$ and all positive integers $n \geqslant 1$

$$P(\|W(t)\|_\alpha \leqslant \varepsilon) \leqslant P(\|W(t)\|_\alpha \leqslant n^{(1-2\alpha)/2}\varepsilon)^n \tag{2.2}$$

This follows because $\{W(t): t \geqslant 0\}$ has stationary independent increments and the rescaling property that as stochastic processes $W(t/n) \stackrel{\mathscr{L}}{=} W(t)/\sqrt{n}$. Hence we have

$$\begin{aligned}
P(\|W(t)\|_\alpha \leqslant \varepsilon) &\leqslant P\left(\sup_{0 \leqslant s < t \leqslant 1-n^{-1}} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \right.\\
&\qquad \left. \leqslant \varepsilon, \quad \sup_{1-n^{-1} \leqslant s < t \leqslant 1} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon \right)\\
&\leqslant P\left(\sup_{0 \leqslant s < t \leqslant 1-n^{-1}} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon\right)\\
&\qquad \times P\left(\sup_{1-n^{-1} \leqslant s < t \leqslant 1} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon\right)\\
&= P\left(\sup_{0 \leqslant s < t \leqslant 1-n^{-1}} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon\right)\\
&\qquad \times P\left(\sup_{0 \leqslant s < t \leqslant n^{-1}} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon\right)\\
&\leqslant P\left(\sup_{0 \leqslant s < t \leqslant 1-n^{-1}} \frac{|W(t)-W(s)|}{|t-s|^\alpha} \leqslant \varepsilon\right)\\
&\qquad \times P(\|W(t)\|_\alpha \leqslant n^{(1-2\alpha)/2}\varepsilon)
\end{aligned}$$

Iterating this procedure, we obtain Eq. (2.2).

The proof now follows a fairly well known argument, see, for example, de Acosta.[1] That is, let

$$l = \varliminf_{\varepsilon \to 0} \varepsilon^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant \varepsilon)$$

$$L = \varlimsup_{\varepsilon \to 0} \varepsilon^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant \varepsilon)$$

Also let $\{a_n\}$, $\{b_n\}$ be two positive sequences such that $a_n \to 0$, $b_n \to 0$, $a_n b_n^{-1} \to \infty$ as $n \to \infty$, and

$$\lim_{n \to \infty} a_n^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant a_n) = l,$$

$$\lim_{n \to \infty} b_n^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant b_n) = L$$

Then by Eq. (2.2),

$$\begin{aligned} &P(\| W(t) \|_\alpha \leqslant b_n) \\ &\quad \leqslant P(\| W(t) \|_\alpha \leqslant [(a_n b_n^{-1})^{2/(1-2\alpha)}]^{(1-2\alpha)/2}\, b_n)^{[(a_n b_n^{-1})^{2/(1-2\alpha)}]} \\ &\quad \leqslant P(\| W(t) \|_\alpha \leqslant a_n)^{[(a_n b_n^{-1})^{2/(1-2\alpha)}]} \end{aligned}$$

where $[x]$ denotes the greatest integer less than x. Hence

$$\begin{aligned} &b_n^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant b_n) \\ &\quad \leqslant a_n^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant a_n) \cdot (b_n a_n^{-1})^{2/(1-2\alpha)} \\ &\qquad \cdot [(a_n b_n^{-1})^{2/(1-2\alpha)}] \end{aligned}$$

impling $L \leqslant l$ and consequently $L = l$.

Now for $\varepsilon > 0$ small and any fixed $x > 0$, there exists an integer $k \geqslant 1$ such that

$$x(k+1)^{-(1-2\alpha)/2} \leqslant \varepsilon < x k^{-(1-2\alpha)/2} \tag{2.3}$$

Thus we have by Eqs. (2.2) and (2.3)

$$\begin{aligned} \varepsilon^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant \varepsilon) &\leqslant \varepsilon^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant x k^{-(1-2\alpha)/2}) \\ &\leqslant \varepsilon^{2/(1-2\alpha)} \log (P(\| W(t) \|_\alpha \leqslant x)^k) \\ &= k \varepsilon^{2/(1-2\alpha)} \log P(\| W(t) \|_\alpha \leqslant x) \\ &\leqslant x^{2/(1-2\alpha)} (k+1)^{-1} k \log P(\| W(t) \|_\alpha \leqslant x) \end{aligned}$$

Hence it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(1-2\alpha)} \log P(\|W(t)\|_\alpha \leqslant \varepsilon) \leqslant x^{2/(1-2\alpha)} \log P(\|W(t)\|_\alpha \leqslant x)$$

which clearly implies the statement of Proposition 2.1.

To prove C_α satisfies Eq. (1.9) we turn to the isomorphism results in Ciesielski.[6] To do this we need some further notation. For $0 \leqslant t \leqslant 1$ we let $h_1(t) = 1$, and

$$h_{2^n+k}(t) = \begin{cases} 2^{n/2} & (2k-2)/2^{n+1} \leqslant t < (2k-1)/2^{n+1} \\ -2^{n/2} & (2k-1)/2^{n+1} \leqslant t < 2k/2^{n+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

for $n = 0, 1,\ldots,$ and $k = 1,\ldots, 2^n$. Thus $\{h_i : i \geqslant 1\}$ is the Haar functions, and for $0 \leqslant t \leqslant 1$ we set

$$\phi_i(t) = \int_0^t h_i(s)\, ds, \qquad i \geqslant 1 \tag{2.5}$$

For $0 < \alpha < 1/2$ and $0 \leqslant t \leqslant 1$ we also define

$$\begin{aligned} \phi_1^\alpha(t) &= \phi_1(t), \qquad h_1^\alpha(t) = h_1(t) \text{ and} \\ \phi_{2^n+k}^\alpha(t) &= 2^{n/2+1-(n+1)\alpha} \phi_{2^n+k}(t) \\ h_{2^n+k}^\alpha(t) &= 2^{(n+1)\alpha-(n/2+1)} h_{2^n+k}(t) \end{aligned} \tag{2.6}$$

for $n = 0, 1,\ldots,$ and $k = 1,\ldots, 2^n$. If $c_0 = C_0(Z^+)$ denotes the space of sequences $\{\xi_i : i \geqslant 1\}$ such that $\lim_{i \to \infty} \xi_i = 0$ and

$$T_\alpha(\{\xi_i\}) = \sum_{i \geqslant 1} \xi_i \phi_i^\alpha \tag{2.7}$$

then Ciesielski[6] [Theorem 2], establishes that T_α is isomorpphism from c_0 onto $H_{\alpha,0}$. Furthermore, if the sup-norm is used on c_0 so that $\|\{\xi_i\}\|_\infty = \sup_{i \geqslant 1} |\xi_i|$ and the α-Hölder norm is used on $H_{\alpha,0}$ then T_α is a bounded linear operator of c_0 onto $H_{\alpha,0}$ with operator norm

$$2/(3(2^\alpha - 1)(2^{1-\alpha} - 1)) \leqslant \|T_\alpha\| \leqslant 2/((2^\alpha - 1)(2^{1-\alpha} - 1)) \tag{2.8}$$

The inverse of T_α from $H_{\alpha,0}$ onto c_0 is given by

$$\xi_i = \int_0^1 h_i^\alpha(s)\, dx(s) \qquad i \geqslant 1 \tag{2.9}$$

where $h_i^\alpha(s)$ is given in Eq. (2.6). Letting T_α^{-1} denote this inverse, Ciesielski[6] [Theorem 2] also establishes that the operator norm of T_α^{-1} is one. Hence if

$$V_\alpha(\delta)=\{x\in H_{\alpha,0}\colon \|T_\alpha^{-1}(x)\|\leqslant\delta\}$$

and $k_\alpha=(2^\alpha-1)(2^{1-\alpha}-1)/2$, then Eq. (2.8) and that $\|T_\alpha^{-1}\|=1$ together imply

$$V_\alpha(k_\alpha\varepsilon)\subseteq\{x\in H_{\alpha,0}\colon \|x\|_\alpha\leqslant\varepsilon\}\subseteq V_\alpha(\varepsilon) \tag{2.10}$$

Thus the constant C_α, $0<\alpha<1/2$, in Eq. (1.8) will satisfy Eq. (1.9) if for any $\delta>0$ and $\varepsilon>0$ small

$$\begin{aligned}\log P(\|T_\alpha^{-1}(W)\|_\infty\leqslant\varepsilon)&\geqslant-(1+\delta)\,2^{-1/(1-2\alpha)}\Gamma_\alpha\varepsilon^{-2/(1-2\alpha)}\\ \log P(\|T_\alpha^{-1}(W)\|_\infty\leqslant\varepsilon)&\leqslant-(1-\delta)\,2^{-2(1-\alpha)/(1-2\alpha)}\Gamma_\alpha\varepsilon^{-2/(1-2\alpha)}\end{aligned} \tag{2.11}$$

To prove Eq. (2.11) we first observe that $\{h_i^\alpha\colon i\geqslant1\}$ are orthogonal functions on [0, 1] with

$$\int_0^1|h_1^\alpha(s)|^2\,ds=1 \qquad\text{and}\qquad \int_0^1|h_{2^n+k}^\alpha(s)|^2\,ds=2^{2((n+1)\alpha-(n/2+1))}$$

Hence

$$\xi_i=\int_0^1 h_i^\alpha(s)\,dW(s)\; i\geqslant1$$

are independent centered Gaussian random variables with variances

$$E(\xi_1^2)=1,\qquad E(\xi_i^2)=2^{-(1-2\alpha)\bar{i}+2(\alpha-1)} \tag{2.12}$$

where $2^{\bar{i}}<i\leqslant2^{\bar{i}+1}$ for $\bar{i}=0,1,\ldots$. Thus

$$P(\|T_\alpha^{-1}(W)\|_\infty\leqslant\varepsilon)=\prod_{i\geqslant1}P(|\xi_i|\leqslant\varepsilon) \tag{2.13}$$

Letting

$$G(t)=(2/\pi)^{1/2}\int_t^\infty e^{-u^2/2}\,du \qquad t\geqslant0 \tag{2.14}$$

we thus have

$$\begin{aligned}P(\|T_\alpha^{-1}(W)\|_\infty\leqslant\varepsilon)&=(1-G(\varepsilon))\prod_{i\geqslant2}P(|\xi_i|\leqslant\varepsilon)\\ &=(1-G(\varepsilon))\prod_{i\geqslant2}(1-G(\varepsilon2^{(1-\alpha)+(1-2\alpha)\bar{i}/2}))\end{aligned} \tag{2.15}$$

Since $2^{\bar{i}} < i \leqslant 2^{\bar{i}+1}$ for $\bar{i} = 0, 1, \ldots$, we have

$$2^{1/2} i^{(1-2\alpha)/2} \leqslant 2^{(1-\alpha)+(1-2\alpha)\bar{i}/2} \leqslant 2^{1-\alpha} i^{(1-2\alpha)/2}$$

for $i \geqslant 2$, and thus

$$\begin{aligned} P(\|T_\alpha^{-1}(W)\|_\infty \leqslant \varepsilon) &\leqslant \prod_{i \geqslant 1} (1 - G(\varepsilon 2^{1-\alpha} i^{(1-2\alpha)/2})) \\ P(\|T_\alpha^{-1}(W)\|_\infty \leqslant \varepsilon) &\geqslant (1 - G(\varepsilon)) \prod_{i \geqslant 2} (1 - G(\varepsilon 2^{1/2} i^{(1-2\alpha)/2})) \end{aligned} \tag{2.16}$$

Now let

$$\Lambda_\beta(\lambda) = \prod_{i \geqslant 1} (1 - G(\lambda i^\beta)) \tag{2.17}$$

for $\lambda > 0$, $\beta > 0$. Hence Eq. (2.11) will follow from Eq. (2.16) with $\beta = (1 - 2\alpha)/2$ and $0 < \alpha < 1/2$, provided we establish the following proposition.

Proposition 2.2. Let $\Lambda_\beta(\lambda)$ be given by Eq. (2.17) with $\lambda > 0$, $\beta > 0$ and assume $G(t)$, $t > 0$, is given by Eq. (2.14). Then as $\lambda \to 0$

$$\log \Lambda_\beta(\lambda) \sim -(2/\pi)^{1/2} \int_0^\infty \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)} \, du \cdot \lambda^{-1/\beta} \tag{2.18}$$

Proof. To prove Eq. (2.18) we first observe

$$\log \Lambda_\beta(\lambda) = \sum_{i \geqslant 1} \log(1 - G(\lambda i^\beta))$$

with $-\infty < \log(1 - G(x)) < 0$ increasing in x for $0 < x < \infty$. Hence

$$\int_1^\infty \log(1 - G(\lambda x^\beta)) \, dx \geqslant \sum_{i \geqslant 1} \log(1 - G(\lambda i^\beta)) \tag{2.19}$$

and by the negativity of $\log(1 - G)$ we also have

$$\sum_{i \geqslant 2} \log(1 - G(\lambda i^\beta)) \geqslant \int_1^\infty \log(1 - G(\lambda x^\beta)) \, dx \tag{2.20}$$

Since

$$\log(1 - G(\lambda)) = \log\left((2/\pi)^{1/2} \int_0^\lambda e^{-u^2/2} \, du\right) \sim \log \lambda \text{ as } \lambda \to 0$$

we thus have from Eqs. (2.19) and (2.20) that Eq. (2.18) holds if

$$\int_1^\infty \log(1-G(\lambda x^\beta))\,dx \sim -(2/\pi)^{1/2}\int_0^\infty \frac{u^{1/\beta}e^{-u^2/2}}{1-G(u)} \times du\cdot\lambda^{-1/\beta} \quad \text{as} \quad \lambda\to 0 \tag{2.21}$$

To prove Eq. (2.21) we integrate by parts and recall $dG(t)/dt = -(2/\pi)^{1/2}\exp(-t^2/2)$ to obtain

$$\int_1^\infty \log(1-G(\lambda x^\beta))\,dx = -\log(1-G(\lambda)) - (2/\pi)^{1/2}\int_\lambda^\infty f^{-1}(u)\frac{e^{-u^2/2}}{1-G(u)}\,du \tag{2.22}$$

where $u=f(x)=\lambda x^\beta$. Hence $f^{-1}(u)=x=(u/x)^{1/\beta}$, and $\log(1-G(\lambda))\sim\log\lambda$ as $\lambda\to 0$, so we have

$$\int_1^\infty \log(1-G(\lambda x^\beta))\,dx \sim -(2/\pi)^{1/2}\int_\lambda^\infty \frac{u^{1/\beta}e^{-u^2/2}}{1-G(u)}\,du\cdot\lambda^{-1/\beta} \quad \text{as} \quad \lambda\to 0 \tag{2.23}$$

Since $1-G(u)=(2/\pi)^{1/2}\int_0^u e^{-t^2/2}\,dt\sim(2/\pi)^{1/2}u$ as $u\to 0$ and $1-G(u)\sim 1$ as $u\to\infty$, Eq. (2.23) and $\beta>0$ yields Eq. (2.21). Thus Proposition 2.2 is proved.

Hence Eq. (2.18) with $\beta=(1-2\alpha)/2, 0<\alpha<1/2$, and Eq. (2.17) together imply Eq. (2.11), so Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 follows along the same lines as those for Theorem 1.1. Hence our first step is to establish analogues of the results in Ciesielski[6] for functions of two variables. For this we need the following notation.

Let $c_0(Z^+\times Z^+)$ denote the sequences $\{\xi_{ij}: i,j\geq 1\}$ such that $\lim_{i\vee j\to\infty}\xi_{ij}=0$ where $i\vee j=\max(i,j)$. We equip $c_0(Z^+\times Z^+)$ with the sup-norm

$$\|\{\xi_{ij}\}\|_\infty = \sup_{i,j\geq 1}|\xi_{ij}| \tag{3.1}$$

and the space $H_{q_\alpha,0}$ with the q_α-norm. Furthermore, we recall the modified Haar and Schauder functions $\{h_i^\alpha: i\geq 1\}$ and $\{\phi_i^\alpha: i\geq 1\}$, respectively, given in Eq. (2.6). Then we have the following proposition.

Proposition 3.1. The linear mapping T_{q_α} defined on $c_0(Z^+ \times Z^+)$ by

$$T_{q_\alpha}(\{\xi_{ij}\}) = \sum_{i,j \geqslant 1} \xi_{ij} \phi_i^\alpha(s)\, \phi_j^\alpha(t) \qquad (s, t) \in I^2 \tag{3.2}$$

maps $c_0(Z^+ \times Z^+)$ in one-to-one fashion onto $H_{q_\alpha,0}$ with the series in Eq. (3.2) converging uniformly on I^2 in an ordering determined in Eq. (3.6). Furthermore, if $c_0(Z^+ \times Z^+)$ has the sup-norm and $H_{q_\alpha,0}$ the q_α-norm, then T_{q_α} is a bounded linear operator between the Banach spaces with operator norm

$$4/(3(2^\alpha - 1)(2^{1-\alpha} - 1))^2 \leqslant \|T_{q_\alpha}\| \leqslant 4/((2^\alpha - 1)(2^{1-\alpha} - 1))^2 \tag{3.3}$$

The inverse operator $T_{q_\alpha}^{-1}$ is given by

$$\xi_{ij} = \iint_{I^2} h_i^\alpha(s)\, h_j^\alpha(s)\, dx^2(s, t) \qquad i, j \geqslant 1 \tag{3.4}$$

and $\|T_{q_\alpha}^{-1}\| = 1$.

Our proof of Proposition 3.1 depends on the fact that $\{\phi_i \phi_j : i, j \geqslant 1\}$ is a Schauder basis for $C_0(I^2)$. This has been proved in a number of papers, but a particularly useful reference is Semadeni,[15] and Ellis and Kuehner[8] provides some additional details. In particular, Semadeni[15] [Theorem 1] implies that for $x \in C_0(I^2)$, the double series

$$\sum_{i,j \geqslant 1} a_{ij} \phi_i(s)\, \phi_j(t) \tag{3.5}$$

converges uniformly to $x(s, t)$ on I^2 when the sequence of products $\phi_i(s)\, \phi_j(t)$ is arranged into a single sequence

$$\tau_k(s, t) = \begin{cases} \phi_{p+1}(s)\, \phi_j(t) \\ \phi_i(s)\, \phi_{p+1}(t) \end{cases} \quad \text{for} \quad \begin{matrix} k = p^2 + i, & 1 \leqslant i \leqslant p \\ k = p^2 + p + i, & 1 \leqslant i \leqslant p + 1 \end{matrix} \tag{3.6}$$

with $p = 0, 1, \ldots$, and the coefficient functionals are given by

$$a_{ij} = a_{ij}(x) = \iint_{I^2} h_i(s)\, h_j(t)\, dx^2(s, t) \qquad i, j \geqslant 1 \tag{3.7}$$

Since the $\{h_i : i \geqslant 1\}$ are the Haar functions, and hence piecewise constant on [0, 1], the Stieljes integral in Eq. (3.7) can be interpreted in the obvious way, i.e. if λ is a constant function, then

$$\int_a^b \int_c^d \lambda\, dx^2(s, t) = \lambda(x(b, d) - x(b, c) - x(a, d) + x(a, b)) \tag{3.8}$$

Then $a_{11}(x)=x(1,1)$ and for $n, m=0, 1,...;\ 1\leqslant k\leqslant 2^n$, and $1\leqslant l\leqslant 2^m$ it is easy to check that

$$a_{2^n+k,2^m+l}(x)=\iint_{I^2} x(s,t)\,d(\mu_{2^n+k}\times\mu_{2^m+k})(s,t) \tag{3.9}$$

where μ_{2^r+j} is the measure on I putting mass $2(2^{r/2})$ at $(2j-1)/2^{r+1}$ and mass $-2^{r/2}$ at each of $(2j-2)/2^{r+1}$ and $(2j)/2^{r+1}$. Hence from Eqs. (3.7)–(3.9) it follows that the sequence $\{a_{ij}(x): i, j\geqslant 1\}$ is uniquely determined by $x\in C_0(I^2)$. Also, the uniform convergence in the ordering given by Eq. (3.6) implies

$$\lim_{N\to\infty}\sup_{(s,t)\in I^2}\left|x(s,t)-\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\,\phi_i(s)\,\phi_j(t)\right|=0 \tag{3.10}$$

as (N, N) is the N^2th term in the ordering.

For each α, $0<\alpha<1/2$, we now have the following lemma.

Lemma 3.1. Let $\{h_i^\alpha: i\geqslant 1\}$ and $\{\phi_i^\alpha: i\geqslant 1\}$ be as in Eq. (2.6), and assume T_{q_α} is defined on $c_0(Z^+\times Z^+)$ by Eq. (3.2). Then $T_{q_\alpha}(\{\xi_{ij}\})\in H_{q_\alpha,0}$ and the series converges uniformly on I^2 in the ordering determined by Eq. (3.6). Also, the right-hand inequality in Eq. (3.3) holds.

Proof. We first show that if $\{\xi_{ij}\}$ is a bounded sequence and $0<\alpha<1/2$, then

$$S=\sup_{(s,t)\in I^2}\sum_{i,j\geqslant 1}|\xi_{ij}|\,|\phi_i^\alpha(s)|\,|\phi_j^\alpha(t)|<\infty \tag{3.11}$$

Recall that the sequence $\{\phi_i^\alpha: i\geqslant 1\}$ consists of nonnegative functions with $\|\phi_1^\alpha\|_\infty=\|\phi_1\|_\infty=1$ and $\|\phi_{2^n+k}^\alpha\|_\infty=2^{-(n+1)\alpha}$. Furthermore, since $\phi_{2^n+k}^\alpha$, $k=1,..., 2^n$, have disjoint support we have

$$\begin{aligned}S\leqslant \|\{\xi_{ij}\}\|_\infty\Bigg(&\sum_{n=0}^{\infty}2^{-(n+1)\alpha}+\sum_{m=0}^{\infty}2^{-(m+1)\alpha}\\ &+\sum_{n,m=0}^{\infty}2^{-(n+1)\alpha}2^{-(m+1)\alpha}\Bigg)<\infty\end{aligned} \tag{3.12}$$

Next we show that if $\{\xi_{ij}\}$ is a bounded sequence, then the series

$$S(\{\xi_{ij}\})(s,t)=\sum_{i,j\geqslant 1}\xi_{ij}\phi_i^\alpha(s)\,\phi_j^\alpha(t) \tag{3.13}$$

is in H_{q_α} for $0<\alpha<1/2$. In view of Eq. (3.11) the series converges uniformly with regard to the ordering determined by Eq. (3.6), and for notational simplicity we denote $S(\{\xi_{ij}\})(s,t)$ by $f(s,t)$. Hence $f\in C_0(I^2)$ and

$$\begin{aligned}\Delta f(s,t,h,h') &= f(s+h,t+h')-f(s,t+h')-f(s+h,t)+f(s,t)\\ &= \sum_{i,j\geqslant 1} \xi_{ij}(\phi_i^\alpha(s+h)-\phi_i^\alpha(s))(\phi_j^\alpha(t+h')-\phi_j^\alpha(t))\end{aligned}$$

Thus

$$\begin{aligned}\Delta f(s,t,h,h') \leqslant \|\{\xi_{ij}\}\|_\infty &\left(|h|+\sum_{n=0}^\infty\sum_{k=1}^{2^n}|\phi_{2^n+k}^\alpha(s+h)-\phi_{2^n+k}^\alpha(s)|\right)\\ &\cdot\left(|h'|+\sum_{m=0}^\infty\sum_{l=1}^{2^m}|\phi_{2^m+k}^\alpha(t+h')-\phi_{2^m+l}^\alpha(t)|\right)\end{aligned}$$

and arguing as in Theorem 1 of Ciesielski[6] we thus have

$$|\Delta f(s,t,h,h')|\leqslant \|\{\xi_{ij}\}\|_\infty\,(|hh'|^\alpha\, 4/((2^\alpha-1)(2^{1-\alpha}-1))^2)$$

Hence

$$q_\alpha(f)\leqslant 4\,\|\{\xi_{ij}\}\|_\infty/((2^\alpha-1)(2^{1-\alpha}-1))^2$$

Thus $f\in H_{q_\alpha}$, and the map T_{q_α} of Eq. (3.2) actually takes $l^\infty(Z^+\times Z^+)$ into H_{q_α}. We also have verified that the right-hand inequality in Eq. (3.3) holds. Furthermore, since finite sums of the $\phi_i^\alpha\phi_j^\alpha$ are in $H_{q_\alpha,0}$, this implies that T_{q_α} maps $c_0(Z^+\times Z^+)$ into $H_{q_\alpha,0}$. Hence Lemma 3.1 is proved.

Our next lemma is the follow.

Lemma 3.2. If $\xi_{ij}=\xi_{ij}(x)$ is given by Eq. (3.4) and $x\in H_{q_\alpha,0}$, then

$$\lim_{i\vee j\to\infty}\xi_{ij}(x)=0 \tag{3.14}$$

Furthermore, the map $\Lambda\colon H_{q_\alpha,0}\to c_0(Z^+\times Z^+)$ given by

$$\Lambda(x)=\{\xi_{ij}(x)\} \tag{3.15}$$

is one-to-one. Hence T_{q_α} is one-to-one and onto $H_{q_\alpha,0}$, and Λ is the inverse of T_{q_α} as defined on $c_0(Z^+\times Z^+)$. Also, the operator norm of $\Lambda=T_{q_\alpha}^{-1}$ from $H_{q_\alpha,0}$ onto $c_0(Z^+\times Z^+)$ is one, and the left-hand inequality in Eq. (3.3) holds.

Proof. If $x \in H_{q_\alpha,0}$, then Eq. (3.14) follows from Eqs. (3.7), (3.9), and (2.6) since they imply

$$\xi_{2^n+k,2^m+l}(x) = 4 \cdot 2^{(n+1)\alpha} 2^{(m+1)\alpha}(I_1 - I_2 - I_3 + I_4)$$

where

$$I_1 = \Delta x((2k-2)/2^{n+1}, (2l-2)/2^{m+1}, 2^{-(n+1)}, 2^{-(m+1)})$$
$$I_2 = \Delta x((2k-1)/2^{n+1}, (2l-2)/2^{m+1}, 2^{-(n+1)}, 2^{-(m+1)})$$
$$I_3 = \Delta x((2k-2)/2^{n+1}, (2l-1)/2^{m+1}, 2^{-(n+1)}, 2^{-(m+1)})$$
$$I_4 = \Delta x((2k-1)/2^{n+1}, (2l-1)/2^{m+1}, 2^{-(n+1)}, 2^{-(m+1)})$$

If $x \in H_{q_\alpha,0}$, we thus have Eq. (3.14), since similar expressions hold for $\xi_{1,2^m+l}(x)$ or $\xi_{2^n+k,1}(x)$. Given the form of $\Lambda(x)$ it is now easy to see Λ is one-to-one and the inverse of T_{q_α} as defined on $c_0(Z^+ \times Z^+)$. Hence T_{q_α} is also one-to-one and onto $H_{q_\alpha,0}$. Furthermore, $T_{q_\alpha}^{-1} = \Lambda$ has operator norm less than or equal to one. To see $\|T_{q_\alpha}^{-1}\| = 1$, we apply $T_{q_\alpha}^{-1}$ to $x(s,t) = st$. Then $\xi_{1,1} = 1$, but $\xi_{i,j}(x) = 0$ when $i \neq 1$ or $j \neq 1$, which yields the result.

To finish the proof of Lemma 3.2 we need to verify the left-hand inequality in Eq. (3.3). This follows as in Ciesielski[6] [Theorem 1].

Combining Lemmas 3.1 3.2, we thus have Proposition 3.1 proved.

To complete the proof of Theorem 1.2 we define

$$U_\alpha(\delta) = \{x \in H_{q_\alpha,0}: \|T_{q_\alpha}^{-1}(x)\|_\infty \leqslant \delta\}$$

and recall $k_\alpha = (2^\alpha - 1)(2^{1-\alpha} - 1)/2$. Then Eq. (3.3) and the operator norm of $T_{q_\alpha}^{-1}$ being one together imply

$$U_\alpha(k_\alpha^2 \varepsilon) \subseteq \{x \in H_{q_\alpha,0}: q_\alpha(x) \leqslant \varepsilon\} \subseteq U_\alpha(\varepsilon) \tag{3.16}$$

Hence Theorem 1.2 will be proved if

$$\log P(\|T_{q_\alpha}^{-1}(W)\|_\infty \leqslant \varepsilon) \approx -\varepsilon^{-2/(1-2\alpha)} \log(1/\varepsilon) \qquad \text{as} \quad \to 0 \tag{3.17}$$

To verify Eq. (3.17) we observe that

$$\xi_{ij} = \iint_{I^2} h_i^\alpha(s)\, h_j^\alpha(t)\, dW^2(s,t) \qquad i, j \geqslant 1$$

are independent centered Gaussian random variables with

$$E(\xi_{11}^2) = 1, \qquad E(\xi_{ij}^2) = 2^{-(1-2\alpha)\bar{i} + 2(\alpha-1)} 2^{-(1-2\alpha)\bar{j} + 2(\alpha-1)} \tag{3.18}$$

where $2^{\bar{i}} < i \leqslant 2^{\bar{i}+1}$ and $2^{\bar{j}} < j \leqslant 2^{\bar{j}+1}$ for $\bar{i}$, $\bar{j} = 0, 1, \ldots$. Hence

$$P(\|T_{q_\alpha}^{-1}(W)\|_\infty \leqslant \varepsilon) = \prod_{i,\, j \geqslant 1} P(|\xi_{ij}| \leqslant \varepsilon) \tag{3.19}$$

Letting $G(t)$ be given by Eq. (2.14) we thus have

$$P(\|T_{q_\alpha}^{-1}(W)\|_\infty \leqslant \varepsilon) = (1 - G(\varepsilon))\, I_1 \cdot I_2 \cdot I_3 \tag{3.20}$$

where

$$\begin{aligned} I_1(\varepsilon) &= \prod_{i \geqslant 2} P(|\xi_{i1}| \leqslant \varepsilon), \\ I_2(\varepsilon) &= \prod_{j \geqslant 2} P(|\xi_{1j}| \leqslant \varepsilon), \\ I_3(\varepsilon) &= \prod_{i,j \geqslant 2} P(|\xi_{ij}| \leqslant \varepsilon) \end{aligned} \tag{3.21}$$

Since $2^{\bar{i}} < i \leqslant 2^{\bar{i}+1}$ and $2^{\bar{j}} < j \leqslant 2^{\bar{j}+1}$ for $\bar{i}$, $\bar{j} = 0, 1, \ldots$, we have

$$2^{1/2} i^{(1-2\alpha)/2} \leqslant 2^{(1-\alpha) + (1-2\alpha)\bar{i}/2} \leqslant 2^{1-\alpha} i^{(1-2\alpha)/2}$$

and

$$2^{1/2} j^{(1-2\alpha)/2} \leqslant 2^{(1-\alpha) + (1-2\alpha)\bar{j}/2} \leqslant 2^{1-\alpha} j^{(1-2\alpha)/2}$$

for i, $j \geqslant 2$ as $0 < \alpha < 1/2$. Thus from Eqs. (2.15) and (2.16), and Proposition 2.2 we have

$$\log P\,(\|T_{q_\alpha}^{-1}(W)\|_\infty \leqslant \varepsilon) \approx -\varepsilon^{-2/(1-2\alpha)} + \log I_3(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{3.22}$$

Hence Theorem 1.2 will be proved if

$$\log I_3(\varepsilon) \approx -\varepsilon^{-2/(1-2\alpha)} \log(1/\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{3.23}$$

and we verify the inequalities following Eq. (1.14). We only verify Eq. (3.23) and the inequalities are easily checked by examining the proof. Letting

$$\Lambda_\beta(\lambda) = \prod_{i,j \geqslant 2} (1 - G(\lambda i^\beta j^\beta)) \tag{3.24}$$

for $\lambda > 0$, $\beta > 0$, we will have Eq. (3.23) be setting $\beta = (1-2\alpha)/2$ and $0 < \alpha < 1/2$, provided we prove the following proposition.

Proposition 3.2. Let $\Lambda_\beta(\lambda)$ be as in Eq. (3.24). Then as $\lambda \to 0$

$$\log \Lambda_\beta(\lambda) \sim -(2/\pi)^{1/2} \frac{1}{\beta} \int_0^\infty \frac{u^{1/2} e^{-u^2/2}}{1 - G(u)}\, du \cdot \lambda^{-1/\beta} \log(1/\lambda) \tag{3.25}$$

Proof. Since $-\infty < \log(1 - G(x) < 0$ is increasing in x for $0 < x < \infty$, it is easy to see from the argument in Proposition 2.2 and the conclusion of Proposition 2.2 that it suffies to prove as $\lambda \to 0$

$$\int_1^\infty \int_1^\infty \log(1 - G(\lambda x^\beta y^\beta))\,dx\,dy \sim -(2/\pi)^{1/2} \frac{1}{\beta} \int_0^\infty \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)} \times du \cdot \lambda^{-1/\beta} \log(1/\lambda). \tag{3.26}$$

Now

$$\int_1^\infty \log(1 - G(\lambda x^\beta y^\beta))\,dx = -\log(1 - G(\lambda y^\beta)) - \lambda^{-1/\beta} \cdot (2/\pi)^{1/2} \times \int_{\lambda y^\beta}^\infty \frac{1}{y} \cdot \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du \tag{3.27}$$

and by Eq. (2.23)

$$\int_1^\infty \log(1 - G(\lambda y^\beta))\,dy \sim -\gamma_\beta \lambda^{-1/\beta} \qquad \text{as} \quad \lambda \to 0 \tag{3.28}$$

where γ_β is a constant independent of λ.

Hence Eq. (3.28) implies

$$\int_1^\infty \int_1^\infty \log(1 - G(\lambda x^\beta y^\beta))\,dx\,dy$$

$$\sim \gamma_\beta \lambda^{-1/\beta} - \lambda^{-1/\beta} \cdot (2/\pi)^{1/2} \int_1^\infty \int_{\lambda y^\beta}^\infty \frac{1}{y} \cdot \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du\,dy. \tag{3.29}$$

Now as $\lambda \to 0$

$$\begin{aligned}
\int_1^\infty \int_{\lambda y^\beta}^\infty \frac{1}{y} \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du\,dy &= \int_\lambda^\infty \int_1^{(u/\lambda)^{1/\beta}} \frac{1}{y} \cdot \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,dy\,du \\
&= \frac{1}{\beta} \int_\lambda^\infty \log(u/\lambda) \cdot \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du \\
&\sim \frac{1}{\beta} \log(1/\lambda) \int_0^\infty \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du
\end{aligned} \tag{3.30}$$

Combining Eqs. (3.29) and (3.30) we thus have

$$\int_1^\infty \int_1^\infty \log(1 - G(\lambda x^\beta y^\beta))\,dx\,dy$$

$$\sim -\frac{1}{\beta} \cdot (2/\pi)^{1/2} \int_0^\infty \frac{u^{1/\beta} e^{-u^2/2}}{1 - G(u)}\,du \cdot \lambda^{-1/\beta} \log(1/\lambda)$$

as $\lambda \to 0$, and Eq. (3.26) holds. Thus Proposition 3.2 is verified and Theorem 1.2 holds.

4. AN APPLICATION TO BROWNIAN MOTION

Let $\{W(t): t \geqslant 0\}$ be a sample continuous Brownian motion on R^1, and assume $H_\mu \subseteq C_0(I)$ is the Hilbert space of absolutely continuous functions on I whose unit ball is the set

$$K = \left\{ f(t) = \int_0^t g(s)\, ds,\ 0 \leqslant t \leqslant 1: \int_0^1 |g(s)|^2\, ds \leqslant 1 \right\} \tag{4.1}$$

Here the inner product norm is given by

$$\|f\|_\mu = \left(\int_0^1 |f'(s)|^2\, ds \right)^{1/2} \qquad f \in H_\mu \tag{4.2}$$

If

$$\eta_n(t) = W(nt)/(2nL_2 n)^{1/2} \qquad 0 \leqslant t \leqslant 1 \tag{4.3}$$

then the functional form of Chung's law of the iterated logarithm given in Csáki,[7] and in more refined form in de Acosta,[1] implies for each f in $C_0(I)$ that with probability one

$$\varliminf_{n \to \infty} L_2 n\, \|\eta_n - f\|_\infty = \begin{cases} \pi/4 \cdot (1 - \|f\|_\mu^2)^{-1/2} & \text{if } \|f\|_\mu < 1 \\ +\infty & \text{otherwise} \end{cases} \tag{4.4}$$

Here and throughout $Lx = \max(1, \log_e x)$, $L_2 x = L(Lx)$ and $L_3 x = L(L_2 x)$.

In view of Theorem 1.1, we can now present the analogue of Eq. (4.4) when the sup-norm is replaced by the α-Hölder norm $\|\cdot\|_\alpha$.

Theorem 4.1. If $0 < \alpha < 1/2$ and C_α is as in Eq. (1.8), then

$$\varliminf_{n \to \infty} (L_2 n)^{1-\alpha}\, \|\eta_n - f\|_\alpha = \begin{cases} 2^{-1/2} C_\alpha^{(1-2\alpha)/2} \cdot (1 - \|f\|_\mu^2)^{-(1-2\alpha)/2} & \text{if } \|f\|_\mu < 1 \\ +\infty & \text{otherwise} \end{cases} \tag{4.5}$$

If $\|f\|_\mu = 1$ and $f(t) = E(W(t)\, h(W))$ where h is a continuous linear functional on $(H_{\alpha,0}, \|\cdot\|_\alpha)$, then with probability one

$$0 < \varliminf_{n \to \infty} (L_2 n)^{2(1-\alpha)/(3-2\alpha)}\, \|\eta_n - f\|_\alpha < \infty \tag{4.6}$$

If $\|f\|_\mu = 1$, but f is not of this form, then with probability one

$$\varliminf_{n\to\infty} (L_2 n)^{2(1-\alpha)/(3-2\alpha)} \|\eta_n - f\|_\alpha = 0 \tag{4.7}$$

To prove Theorem 4.1 we first establish the following proposition for i.i.d. samples of Brownian motion. The remainder of the proof is handled by fairly standard rescaling arguments.

Proposition 4.1. Let $W_1, W_2, \ldots$ be i.i.d. copies of W and let $0 < \alpha < 1/2$. Then with probability one

$$\varliminf_{n\to\infty} (Ln)^{1-\alpha} \| W_n/(2Ln)^{1/2} - f\|_\alpha$$
$$= \begin{cases} 2^{-1/2} C_\alpha^{(1-2\alpha)/2} \cdot (1 - \|f\|_\mu^2)^{-(1-2\alpha)/2} & \text{if } \|f\|_\mu < 1 \\ +\infty & \text{otherwise} \end{cases} \tag{4.8}$$

Furthermore, if $\|f\|_\mu = 1$ and $f(t) = E(W(t)\, h(W))$ where h is a continuous linear functional on $(H_{\alpha,0}, \|\cdot\|_\alpha)$, then with probability one

$$0 < \varliminf_{n\to\infty} (Ln)^{2(1-\alpha)/(3-2\alpha)} \| W_n/(2Ln)^{1/2} - f\|_\alpha < \infty \tag{4.9}$$

If $\|f\|_\mu = 1$, but f is not of this form, then with probability one

$$\varliminf_{n\to\infty} (Ln)^{2(1-\alpha)/(3-2\alpha)} \| W_n/(2Ln)^{1/2} - f\|_\alpha = 0 \tag{4.10}$$

Proof of Proposition 4.1. If $\|f\|_\mu < 1$, then the corresponding part of Eq. (4.8) holds by applying the Borel-Cantelli lemma and the following result.

Lemma 4.1. If $f \in H_\mu$, $r > 0$, and $0 < \alpha < 1/2$, then

$$\lim_{\lambda\to\infty} \lambda^{-2} \log P(\| W - \lambda f\|_\alpha \leqslant \lambda^{-(1-2\alpha)} r) = -\tfrac{1}{2}\|f\|_\mu^2 - C_\alpha r^{-2/(1-2\alpha)} \tag{4.11}$$

where C_α is given in Eq. (1.8).

Proof. The proof of Lemma 4.1 follows that of Theorem 3.3 in de Acosta.[1] In particular, it requires the Cameron-Martin translation formula which asserts that for $f \in H_\mu$

$$P(\| W - \lambda f\|_\alpha \leqslant \tau)$$
$$= \exp\left(-\frac{\lambda^2}{2}\|f\|_\mu^2\right) \int_{\{x:\,\|x\|_\alpha \leqslant \tau\}} \exp(-\lambda\langle x, f\rangle^\sim)\, d\mu(x) \tag{4.12}$$

The measure μ is Wiener measure and $\langle x, f\rangle^{\sim}$ denotes a "stochastic inner product" which is $N(0, \|f\|_\mu^2)$ and is such that $\langle x, f\rangle^{\sim} = f(x)$ if f is continuous and linear. Hence Jensen's inequality and the symmetry of the set $\{x: \|x\|_\alpha \leqslant \tau\}$ imply

$$P(\|W-\lambda f\|_\alpha \leqslant \tau) \geqslant \exp\left(-\frac{\lambda^2}{2}\|f\|_\mu^2\right) \cdot P(\|W\|_\alpha \leqslant \tau)$$

Setting $\tau = -\lambda^{1-2\alpha} r$ and recalling Eq. (1.8) we have

$$\varliminf_{\lambda \to \infty} \lambda^{-2} \log P(\|W-\lambda f\|_\alpha \leqslant \lambda^{-(1-2\alpha)} r) \geqslant -\tfrac{1}{2}\|f\|_\mu^2 - C_\alpha r^{-2/(1-2\alpha)} \tag{4.13}$$

Since $f \in H_\mu$, we can write $f(\cdot) = \sum_{j \geqslant 1} \int_0^1 h_j(s)\, df(s)\, \phi_j(\cdot)$ where the $\{\phi_j\}$ and $\{h_j\}$ are the Schauder and Haar functions, respectively, and $\{\phi_j\}$ are C.O.N.S in H_μ. Hence given $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $\|f - \sum_{j=1}^N \int_0^1 h_j(s)\, df(s)\, \phi_j\|_\mu^2 < \varepsilon$. Letting $g = \sum_{j=1}^N \int_0^1 h_j(s)\, df(s)\, \phi_j$ we have

$$P(\|W-\lambda f\|_\alpha \leqslant \tau) \leqslant \exp\left(-\frac{\lambda^2}{2}\|f\|_\mu^2 + \lambda \sup_{x: \|x\|_\alpha \leqslant \tau} |\langle x, g\rangle^{\sim}|\right)$$
$$\times \int_{\{x: \|x\|_\alpha \leqslant \tau\}} \exp(-\lambda\langle x, f-g\rangle^{\sim})\, d\mu(x) \tag{4.14}$$

Now $\langle x, g\rangle^{\sim} = \sum_{j=1}^N \int_0^1 h_j(s)\, df(s) \int_0^1 h_j(s)\, dx(s)$ (see, for example, Kuelbs *et al.*[11] for details), so

$$\sup_{x: \|x\|_\alpha \leqslant \tau} |\langle x, g\rangle^{\sim}| \leqslant \sum_{j=1}^N \left|\int_0^1 h_j(s)\, df(s)\right| \cdot \sup_{x: \|x\|_\alpha \leqslant \tau} \left|\int_0^1 h_j(s)\, dx(s)\right| \leqslant M_N \tau$$

and we also have

$$\int_{\{x: \|x\|_\alpha \leqslant \tau\}} \exp(-\lambda\langle x, f-g\rangle^{\sim})\, d\mu(x)$$
$$= \mu(x: \|x + (f-g)\|_\alpha \leqslant \tau) \exp\left(\frac{\lambda^2}{2}\|f-g\|_\mu^2\right)$$
$$\leqslant \mu(x: \|x\|_\alpha \leqslant \tau) \exp\left(\frac{\lambda^2}{2}\|f-g\|_\mu^2\right)$$

by the Cameron-Martin formula since $\mu(x+U) \leqslant \mu(U)$ for every convex symmetric Borel measurable set U. Hence

$$P(\|W-\lambda f\|_\alpha \leqslant \tau) \leqslant \exp\left(-\frac{\lambda^2}{2}\|f\|_\mu^2 + \frac{\lambda^2}{2}\|f-g\|_\mu^2 + \lambda M_N \tau\right) P(\|W\|_\alpha \leqslant \tau) \tag{4.15}$$

Setting $\tau=\lambda^{-(1-2\alpha)}r$ and recalling $\|f-g\|_\mu^2<\varepsilon$ we have

$$\overline{\lim_{\lambda\to\infty}}\ \lambda^{-2}\log P(\|W-\lambda f\|_\alpha\leqslant\lambda^{-(1-2\alpha)}r)\leqslant-\|f\|_\mu^2/2+\varepsilon/2-C_\alpha r^{-2/(1-2\alpha)} \tag{4.16}$$

Since $\varepsilon>0$ is arbitrary, Eqs. (4.16) and (4.13) combine to yield Eq. (4.11), so the lemma is proved.

Hence we have Eq. (4.8) when $\|f\|_\mu<1$. If $\|f\|_\mu\geqslant 1$ the same argument yields the latter half of (4.8), but we also can see this as follows. That is, if $\|f\|_\mu=1$, then Theorem 1* of Kuelbs et al.[11] implies that with probability one

$$\underline{\lim}_{n\to\infty}\ d_n^{-1}\ \|W_n/(2Ln)^{1/2}-f\|_\alpha>0,$$

where $d_n^{-1}\cdot\rho_n=(Ln)^{1-\alpha}$ and $\rho_n\to\infty$. Hence the latter half of (4.8) holds if $\|f\|_\mu=1$. If $\|f\|_\mu>1$, then the result follows immediately since we have for all $\varepsilon>0$ that

$$P(W_n/(2Ln)^{1/2}\in K+\varepsilon U\text{ eventually})=1 \tag{4.17}$$

when $U=\{x\colon\|x\|_\alpha\leqslant 1\}$. See Theorem 2.1 and remark (C) following Theorem 2.1 in Goodman and Kuelbs[10] for details regarding Eq. (4.17). Furthermore, note that $(H_{\alpha,0},\|\cdot\|_\alpha)$ is a separable Banach space which supports the Wiener measure μ and hence it is well known that the pair H_μ and $(H_{\alpha,0},\|\cdot\|_\alpha)$ form an abstract Wiener space. Hence with μ probability one we have $\lim_{d\to\infty}\|x-\Pi_d(x)\|_\alpha=0$, or, equivalently, that $\lim_{d\to\infty}\|Q_d(x)\|_\alpha=0$. Here

$$\Pi_d(x)=\sum_{j=1}^d\langle x,\alpha_j\rangle^\sim\alpha_j\qquad\text{and}\qquad Q_d(x)=x-\Pi_d(x)\ d\geqslant 1$$

where $\{\alpha_j\}$ is an arbitrary C.O.N.S. in H_μ(see, for example, Fernique[9]). Thus Theorem 2.1 in Goodman and Kuelbs[10] readily applies.

Hence it remains to verify Eqs. (4.9) and (4.10). In view of the isomorphisms T_α and T_α^{-1} defined in Eqs. (2.7) and (2.9), respectively, and Proposition 1.2.4 of Fernique,[9] it suffices to prove that with probability one

$$0<\underline{\lim}_{n\to\infty}\ (Ln)^{2(1-\alpha)/(3-2\alpha)}\ \|X_n/(2Ln)^{1/2}-T_\alpha^{-1}(f)\|_\infty<\infty \tag{4.18}$$

when $f=E(Wh(W))$ and h is a continuous functional on $(H_{\alpha,0},\|\cdot\|_\alpha)$, and

$$\underline{\lim}_{n\to\infty}\ (Ln)^{2(1-\alpha)/(3-2\alpha)}\ \|X_n/(2Ln)^{1/2}-T_\alpha^{-1}(f)\|_\infty=0 \tag{4.19}$$

when f is not of the this form. In Eqs. (4.18) and (4.19) $X_n = T_\alpha^{-1}(W_n)$, $n \geqslant 1$, are i.i.d. independent coordinate centered Gaussian vectors with values in c_0 and $g = T_\alpha^{-1}(f)$ satisfies $\|g\|_\nu = 1$ where ν is the Gaussian measure on c_0 induced by $X = T_\alpha^{-1}(W)$. Here $\|g\|_\nu = 1$ follows as in Fernique,[9] but also it is easy to check this directly. That is, if $f = E(Wh(W))$ where h is continuous on $(H_{\alpha,0}, \|\cdot\|_\alpha)$, then $g = T_\alpha^{-1}(f) = E(T_\alpha^{-1}(W)\,h(W)) = E(Xh \circ T_\alpha(X))$ and $h \circ T_\alpha$ is continuous on c_0. Thus

$$\|g\|_\nu^2 = E((h \circ T_\alpha(X))^2) = E(h^2(W)) = \|f\|_\mu^2 = 1$$

as claimed. Now we can apply the results in Kuelbs *et al.*[11] to this problem. That is, Theorem 1 of Kuelbs *et al.*[11] imply Eqs. (4.18) and (4.19) since Eq. (2.11) and Proposition 3.2 imply

$$\log P(\|X\| \leqslant \varepsilon) \approx -\varepsilon^{2/(1-2\alpha)} \qquad \text{as} \quad \varepsilon \to 0$$

and solving for $d = d(n)$ in Eq. (3.3) of Kuelbs *et al.*[11] we have $d(n) \approx (Ln)^{-2(1-\alpha)/(3-2\alpha)}$. Thus Proposition 4.1 is proved.

To complete the proof of Theorem 4.1 we now apply some rescaling arguments. Hence we present the following lemmas. The first follows calculations similar to those in Lemma 3.4 in Baldi *et al.*[5]

Lemma 4.2. If $f \in H_\mu$ and $g(\cdot) = f(\lambda(\cdot))$ on I, and $0 < \lambda < 1$, then for $0 < \alpha < 1/2$

$$\|f - g\|_\alpha \leqslant 2\,|1-\lambda|^{1/2-\alpha}\,\|f\|_\mu \tag{4.20}$$

Proof. Since $g(\cdot) = f(\lambda(\cdot))$,

$$\|f - g\|_\alpha = \sup_{0 \leqslant s \leqslant t \leqslant 1} \frac{|(f(t) - f(\lambda t)) - (f(s) - f(\lambda s))|}{|t-s|^\alpha} \tag{4.21}$$

If $0 \leqslant s < t \leqslant 1$, then

$$\begin{aligned} |(f(t) - f(\lambda t)) - (f(s) - f(\lambda s))| &= \left| \int_{s \vee \lambda t}^{t} f'(u)\,du - \int_{\lambda s}^{s \wedge \lambda t} f'(u)\,du \right| \\ &\leqslant \gamma(s, t, \lambda)\,\|f\|_\mu \end{aligned} \tag{4.22}$$

where

$$\gamma(s, t, \lambda) = (|t - (s \vee \lambda t)|^{1/2} + |(s \wedge \lambda t) - \lambda s|^{1/2})$$

If $\lambda t \leqslant s$, $0 \leqslant s < t \leqslant 1$, then $s \vee \lambda t = s$, $s \wedge \lambda t = \lambda t$, so

$$\gamma(s, t, \lambda) = |t-s|^{1/2}\,(1 + \lambda^{1/2}) \tag{4.23}$$

and

$$|t-s|^{1/2-\alpha} = t^{1/2-\alpha}\,|1 - s/t|^{1/2-\alpha} \leqslant |1-\lambda|^{1/2-\alpha} \tag{4.24}$$

Similarly, if $s \leqslant \lambda t$, $0 \leqslant s < t \leqslant 1$, then $s \vee \lambda t$, $s \wedge \lambda t = s$,

$$\gamma(s, t, \lambda) = |t - \lambda t|^{1/2} + |s - \lambda s|^{1/2} \leqslant 2t^{1/2} \, |1 - \lambda|^{1/2} \tag{4.25}$$

and

$$|t - s|^{-\alpha} = t^{-\alpha} \, |1 - s/t|^{-\alpha} \leqslant t^{-\alpha} \, |1 - \lambda|^{-\alpha} \tag{4.26}$$

Combining Eq. (4.21) to Eq. (4.26) and that $0 < \alpha < 1/2$, we obtain Eq. (4.20). Hence the lemma is proved.

Lemma 4.3. Let m, n, r be nonnegative integers with $m \leqslant n \leqslant r$ and $a_n = (2nL_2 n)^{1/2}$ for $n \geqslant 1$. Then for all $f \in H_\mu$ and $1/2 < p < 1$

$$\begin{aligned}
&(L_2 n)^p \, \| W(n(\cdot))/a_n - f \|_\alpha \\
&\quad \geqslant \left(\frac{n}{m}\right)^\alpha \left(\frac{m}{r}\right)^{1/2} (L_2 m)^p \, \| W(m(\cdot))/a_m - f \|_\alpha \\
&\quad\quad - 2\left(\frac{r}{m}\right)^\alpha (L_2 r)^p \left| 1 - \frac{m}{r} \right|^{1/2 - \alpha} \| f \|_\mu \\
&\quad\quad - \left(\frac{r}{m}\right)^\alpha (L_2 r)^p \left(1 - \frac{a_m}{a_r}\right) \| f \|_\alpha
\end{aligned} \tag{4.27}$$

Proof. This is a slight modification of Lemma 5.3 in de Acosta[1] applicable to the α-Hölder norm rather than the sup-norm. Since $W(n(\frac{m}{n}(\cdot)) = W(m(\cdot))$ we have

$$\begin{aligned}
(L_2 n)^p \, \| W(n(\cdot))/a_n - f \|_\alpha &\geqslant \left(\frac{n}{m}\right)^\alpha \frac{(L_2 n)^p}{a_n} \left\| W(m(\cdot)) - a_n f\left(\frac{m}{n}(\cdot)\right) \right\|_\alpha \\
&\geqslant \left(\frac{n}{m}\right)^\alpha \frac{(L_2 r)^p}{a_r} \, \| W(m(\cdot)) - a_n \, g(\cdot) \|_\alpha
\end{aligned} \tag{4.28}$$

where $g(\cdot) = f(\frac{m}{n}(\cdot))$. Now

$$\| W(m(\cdot)) - a_n \, g(\cdot) \|_\alpha \geqslant \| W(m(\cdot)) - a_m f \|_\alpha - a_r \, \| f - g \|_\alpha - (a_r - a_m) \, \| f \|_\alpha \tag{4.29}$$

and since $f \in H_\mu$ Lemma 4.2 implies

$$\| f - g \|_\alpha \leqslant 2 \, |1 - m/r|^{1/2 - \alpha} \, \| f \|_\mu \tag{4.30}$$

Combining Eqs. (4.28)–(4.30) we get Eq. (4.27). Hence the lemma is proved.

Lemma 4.4. Let $n_0 = 0$, $n_r = \exp(r/(Lr)^\tau)$ and $I(r) = [n_{r-1}, n_r)$ for $r \geqslant 1$. Let m_r denote the smallest integer greater than or equal to n_{r-1} and p_r denote the largest integer less than or equal to n_r. Then for $1/2 < p < 1$, $0 < \alpha < 1/2$, and $\tau > 2/(1-2\alpha)$, we have

$$\varliminf_{r \to \infty} \inf_{n \in I(r)} (L_2 n)^p \, \|\eta_n - f\|_\alpha \geqslant \varliminf_{r \to \infty} (L_2 m_r)^p \, \|\eta_{m_r} - f\|_\alpha \tag{4.31}$$

Proof. In view of Lemma 4.3, this follows since $\lim_{r \to \infty} n_{r-1}/n_r = 1$,

$$\varlimsup_{r \to \infty} (L_2 p_r)^p \, |1 - m_r/p_r|^{1/2-\alpha} = 0, \quad \text{and} \quad \varlimsup_{r \to \infty} (L_2 p_r)^p \, (1 - a_{m_r}/a_{p_r}) = 0$$

when $\tau > 2/(1-2\alpha)$.

Proof of Theorem 4.1. If $W_r = W(m_r(\cdot))/m_r^{1/2}$, then $W_1, W_2, \ldots$, are identically distributed copies of Brownian motion. If $\|f\|_\mu < 1$, then the proof of Proposition 4.1 shows that with probability one

$$\begin{aligned} &\varliminf_{r \to \infty} (Lr)^{1-\alpha} \, \|W_r/(2Lr)^{1/2} - f\|_\alpha \\ &\qquad \geqslant 2^{-1/2} C_\alpha^{(1-2\alpha)/2} \cdot (1 - \|f\|_\mu^2)^{-(1-2\alpha)/2} \end{aligned} \tag{4.32}$$

Independence is not required here since this depends on the convergence part of the Borel-Cantelli lemma. Now $Lr \sim L_2 m_r$ and since $0 < \alpha < 1/2$ we have from Eq. (4.17) that with probability one

$$\varlimsup_{r \to \infty} (Lr)^{1-\alpha} \left\| W_r \cdot \left(\frac{1}{(2Lr)^{1/2}} - \frac{1}{(2L_2 m_r)^{1/2}} \right) \right\|_\alpha = 0 \tag{4.33}$$

Hence Eqs. (4.31)–(4.33) imply with probability one

$$\varliminf_{n \to \infty} (L_2 n)^{1-\alpha} \, \|\eta_n - f\|_\alpha \geqslant 2^{-1/2} C_\alpha^{(1-2\alpha)/2} \cdot (1 - \|f\|_\mu^2)^{-(1-2\alpha)/2} \tag{4.34}$$

The reverse inequality in Eq. (4.34) when $\|f\|_\mu < 1$ is similar to the argument starting with Eq. (4.38) to establish the right-hand side of Eq. (4.6). In Eq. (4.6), $\|f\|_\mu = 1$ and this is more delicate, so we only include the details of that argument.

Similarly, if $\|f\|_\mu > 1$, then Eq. (4.17) and the previous argument implies that with probability one

$$\varliminf_{n \to \infty} (L_2 n)^{1-\alpha} \, \|\eta_n - f\|_\alpha = \infty \tag{4.35}$$

It is also the case that if $\|f\|_\mu = 1$, then the proof of Proposition 4.1 implies with probability one that

$$\underline{\lim}_{r\to\infty} (Lr)^{2(1-\alpha)/(3-2\alpha)} \| W_r/(2Lr)^{1/2} - f\|_\alpha > 0 \tag{4.36}$$

when $f(t) = E(W(t)\,h(W))$ and h is a continuous linear functional on $(H_{\alpha,0}, \|\cdot\|_\alpha)$. Hence Eqs. (4.33), (4.31), and (4.36) imply the lim inf in Eq. (4.6) is positive. To verify (4.7) we need to prove that if $\|f\|_\mu = 1$, but f is not of the above form, then we also have with probability one that

$$\underline{\lim}_{r\to\infty} (Lr)^{2(1-\alpha)/(3-2\alpha)} \| W(n_r(\cdot))/(2n_r L_2 n_r)^{1/2} - f\|_\alpha = 0 \tag{4.37}$$

for the sequence $n_r = \exp(rLr)$. This requires some independence and so also does the argument to establish the right hand side of (4.6). We turn to the proof that the lim inf in (4.6) is finite as the proof of (4.37) will follow by a similar argument.

Let $n_0 = 1$, $n_r = \exp(rLr)$ for $r \geqslant 1$. Let

$$\begin{aligned} W_r(t) = (&W((n_r - n_{r-1})\,t + n_{r-1}) \\ &- W(n_{r-1}))/(n_r - n_{r-1})^{1/2} \qquad 0 \leqslant t \leqslant 1 \end{aligned} \tag{4.38}$$

for $r \geqslant 1$. Then $W_1, W_2,...,$ are i.i.d. copies of Brownian motion for $0 \leqslant t \leqslant 1$ and Proposition 4.1 implies with probability one

$$\underline{\lim}_{r\to\infty} (Lr)^{2(1-\alpha)/(3-2\alpha)} \| W_r/(2Lr)^{1/2} - f\|_\alpha < \infty \tag{4.39}$$

when $f(t) = E(W(t)\,h(W))$ for h a continuous linear functional on $(H_{\alpha,0}, \|\cdot\|_\alpha)$. Hence it is suffices to show

$$\overline{\lim}_{r\to\infty} (Lr)^{2(1-\alpha)/(3-2\alpha)} \| W_r/(2Lr)^{1/2} - W(n_r(\cdot))/(2n_r L_2 n_r)^{1/2}\|_\alpha = 0 \tag{4.40}$$

with probability one. Since $n_r = \exp(rLr)$ we have

$$\overline{\lim}_{r\to\infty} \frac{(Lr)^{3/2}}{L_2 r} \left| \frac{1}{(Lr)^{1/2}} - \frac{1}{(L_2 n_r)^{1/2}} \right| < \infty$$

and hence Eq. (4.17) implies that Eq. (4.40) will hold if

$$\overline{\lim}_{r\to\infty} (\mathrm{Lr})^{2(1-\alpha)/(3-2\alpha)} \| W_r/(2Lr)^{1/2} - W(n_r(\cdot))/(2n_r Lr)^{1/2}\|_\alpha = 0 \tag{4.41}$$

with probability one. Since

$$\overline{\lim}_{r\to\infty} r n_r^{1/2} \left| \frac{1}{(n_r - n_{r-1})^{1/2}} - \frac{1}{n_r^{1/2}} \right| < \infty$$

it follows from Eq. (4.17) that

$$\overline{\lim_{r\to\infty}}\ (Lr)^{2(1-\alpha)/(3-2\alpha)} \left\| \frac{W_r(\cdot)\cdot(n_r-n_{r-1})^{1/2}}{(2Lr)^{1/2}} \left(\frac{1}{(n_r-n_{r-1})^{1/2}} - \frac{1}{n_r^{1/2}} \right) \right\|_\alpha = 0 \tag{4.42}$$

and hence it is suffices to show that with probability one

$$\overline{\lim_{r\to\infty}}\ (Lr)^{2(1-\alpha)/(3-2\alpha)} \times \left\| \frac{W((n_r-n_{r-1})(\cdot)+n_{r-1})-W(n_{r-1})-W(n_r(\cdot))}{(2n_rLr)^{1/2}} \right\|_\alpha = 0 \tag{4.43}$$

Thus for $b>0$, scaling Brownian motion implies

$$\begin{aligned} &P\left(\left\| \frac{W((n_r-n_{r-1})(\cdot)+n_{r-1})-W(n_r(\cdot))}{n_r^{1/2}} \right\|_\alpha \geqslant b \right) \\ &\quad = P\left(\left\| W\left(\left(1-\frac{n_{r-1}}{n_r}\right)(\cdot)+\frac{n_{r-1}}{n_r} \right) - W(\cdot) \right\|_\alpha \geqslant b \right) \\ &\quad = P(W(\cdot)\in A(b,r)) \end{aligned}$$

where

$$A(b,r)=\left\{ x\in H_{\alpha,0}: \left\| x\left(\left(1-\frac{n_{r-1}}{n_r}\right)(\cdot)+\frac{n_{r-1}}{n_r} \right) - x(\cdot) \right\|_\alpha \geqslant b \right\}$$

Now $A(b,r)$ is closed in $(H_{\alpha,0}, \|\cdot\|_\alpha)$, so by the large deviation results in Baldi *et al.*,[(5)] we have

$$\log P(A(b,r)) \leqslant -\tfrac{1}{2} \inf_{f\in A(b,r)} \int_0^1 |f'(s)|^2\, ds \tag{4.44}$$

If $f\in H_\mu$ and $f\in A(b,r)$, then a calculation similar to that used for the proof of Lemma 4.2 implies

$$\|f\|_\mu \geqslant 2^{-1/2} b \tau_r^{\alpha-1/2} \tag{4.45}$$

where $\tau_r = n_{r-1}/n_r \leqslant r^{-1}$. Hence for $\varepsilon>0$, Eqs. (4.44) and (4.45) imply

$$\begin{aligned} &P(\|W((n_r-n_{r-1})(\cdot)+n_{r-1})-W(n_{r-1})-W(n_r(\cdot))\|_\alpha \\ &\quad \geqslant n_r^{1/2}\varepsilon(Lr)^{1/2-2(1-\alpha)/(3-2\alpha)}) \\ &\quad \leqslant \exp\left(-\frac{\varepsilon^2}{4}(Lr)^{-(1-2\alpha)/(3-2\alpha)} r^{1-2\alpha} \right) \end{aligned} \tag{4.46}$$

Now the terms in Eq. (4.46) are summable, and since $\varepsilon>0$ is arbitrary Eq. (4.43) holds. Thus Eq. (4.6) is finite. By the same sort of argument we now can also establish (4.37), and hence Theorem 4.1 is proved.

5. AN APPLICATION TO THE BROWNIAN SHEET

If $\{W(s, t): s, t \geqslant 0\}$ is a sample continuous Brownian sheet, then the analogue of K is the set

$$K_2 = \left\{ f(s, t) = \int_0^s \int_0^t g(u, v)\, du\, dv, 0 \leqslant s, t \leqslant 1: \iint_{I^2} g^2(u, v)\, du\, dv \leqslant 1 \right\} \tag{5.1}$$

and the relevant inner product norm on the Hilbert space $H_{\mu, 2}$ with unit ball K_2 is given by

$$\|f\|_{\mu,2} = \left(\iint_{I^2} g(u, v)\, du\, dv \right)^{1/2} \tag{5.2}$$

when $f(s, t) = \int_0^s \int_0^t g(u, v)\, du\, dv$, $(s, t) \in I^2$. See, for example Park[14] [Theorem 6]. The analogue of Theorem 4.1 is the following Theorem.

Theorem 5.1. Let $\{W(s, t): s, t \geqslant 0\}$ be a sample continuous Brownian sheet and set

$$\eta_{n,2}(s, t) = W(ns, nt)/(n(2L_2 n)^{1/2}) \qquad (s, t) \in I^2 \tag{5.3}$$

Let q_α denote the α-Hölder norm in Theorem 1.2 and assume $0 < \alpha < 1/2$. Then with probability one

$$0 < \varliminf_{n \to \infty} (L_2 n)^{1-\alpha} (L_3 n)^{-(1-2\alpha)/2} q_\alpha(\eta_{n,2} - f) < \infty \tag{5.4}$$

if $\|f\|_{\mu,2} < 1$ and it is infinity otherwise. If $\|f\|_{\mu,2} = 1$ and $f(s, t) = E(W(s, t)\, h(W))$ where h is a continuous linear functional on $(H_{q_\alpha, 0}, q_\alpha)$, then with probability one

$$0 < \varliminf_{n \to \infty} (L_2 n)^{2(1-\alpha)/(3-2\alpha)} (L_3 n)^{-(1-2\alpha)/(3-2\alpha)} q_\alpha(\eta_{n,2} - f) < \infty \tag{5.5}$$

If $\|f\|_{\mu,2} = 1$, but f is not of the form previously indicated, then with probability one

$$\varliminf_{n \to \infty} (L_2 n)^{2(1-\alpha)/(3-2\alpha)} (L_3 n)^{-(1-2\alpha)/(3-2\alpha)} q_\alpha(\eta_{n,2} - f) = 0 \tag{5.6}$$

The proof of Eq. (5.4) can be obtained by establishing a companion result to Proposition 4.1.

Proposition 5.1. Let $W_1, W_2,...,$ be i.i.d. copies of a sample continuous Brownian sheet and let $0<\alpha<1/2$. Then with probability one

$$\varliminf_{n\to\infty} (Ln)^{1-\alpha}(L_2n)^{-(1-2\alpha)/2}\, q_\alpha(W_n/(2Ln)^{1/2}-f)<\infty \tag{5.7}$$

if $\|f\|_{\mu,2}<1$ and it is infinity otherwise. If $\|f\|_{\mu,2}=1$ and $f(s,t)=E(W(s,t)\,h(W))$ where h is a continuous linear functional on $(H_{q_\alpha,0}, q_\alpha)$, then with probability one

$$0<\varliminf_{n\to\infty} (Ln)^{2(1-\alpha)/(3-2\alpha)}(L_2n)^{-(1-2\alpha)/(3-2\alpha)}q_\alpha(W_n/(2Ln)^{1/2}-f)<\infty \tag{5.8}$$

If $\|f\|_{\mu,2}=1$, but f is not of the form indicated, then with probability one

$$\varliminf_{n\to\infty} (Ln)^{2(1-\alpha)/(3-2\alpha)}(L_2n)^{-(1-2\alpha)/(3-2\alpha)}\, q_\alpha(W_n/(2Ln)^{1/2}-f)=0 \tag{5.9}$$

Proof of Proposition 5.1. If $\|f\|_{\mu,2}\leqslant 1$, then Eq. (5.7) follows by applying Theorem 1* of Kuelbs *et al.*[11] If $\|f\|_{\mu,2}<1$, then we could also obtain this by proving an analogue to Lemma 4.1, but since we do not known the constant analogous to C_α we chose to use Theorem 1*. If $\|f\|_{\mu,2}>1$, then by applying Theorem 2.1 in Goodman and Kuelbs[10] as indicated following Eq. (4.17), we have with probability one

$$\varliminf_{n\to\infty} q_\alpha(W_n/(2Ln)^{1/2}-f)>0 \tag{5.10}$$

Hence if $\|f\|_{\mu,2}>1$, the lim inf in Eq. (5.7) is infinity as claimed.

To prove Eqs. (5.8) and (5.9) we use the isomorphism between $(c_0(Z^+\times Z^+), \|\cdot\|_\infty)$ and $(H_{q_\alpha,0}, q_\alpha)$ discussed in the proof of Theorem 1.2, and Theorem 1 in Kuelbs *et al.*[11] for centered independent coordinate Gaussian measures on $c_0(Z^+\times Z^+)$. Combining these results with Eqs. (3.16), (3.19), and Eq. (3.25) we get Eqs. (5.8) and (5.9). Hence Proposition 5.1 is verified.

To finish the proof of Theorem 5.1 we apply the rescaling arguments of Section 4 to the setting of the Brownian sheet.

Lemma 5.1. If $f\in H_{\mu,2}$ with $f(s,t)=\int_0^s\int_0^t g(u,v)\,du\,dv$ and $f^*(\cdot)=f(\lambda(\cdot))$ on I^2, and $1/2<\lambda<1$, then for $0<\alpha<1/2$

$$q_\alpha(f-f^*)\leqslant 4\,|1-\lambda|^{1/2-\alpha}\,\|f\|_{\mu,2} \tag{5.11}$$

Proof. Since $f^*(\cdot)=f(\lambda(\cdot))$,

$$q_\alpha(f-f^*)=\sup_{\substack{(s,t)\in I^2\\(s+h,t+h')\in I^2\\h,h'>0}}\frac{|\Delta f(s,t,h,h')-\Delta f^*(s,t,h,h')|}{(hh')^\alpha} \tag{5.12}$$

where $\Delta f(s, t, h, h')$ is given in Eq. (1.4). If $(s, t) \in I^2$, $(s+h, t+h') \in I^2$, $h, h' > 0$, then

$$\begin{aligned}
&|\Delta f(s, t, h, h') - \Delta f^*(s, t, h, h')| \\
&\quad = \left| \int_s^{s+h} \int_t^{t+h'} g(u, v)\, du\, dv - \int_{\lambda s}^{\lambda(s+h)} \int_{\lambda t}^{\lambda(t+h')} g(u, v)\, du\, dv \right| \\
&\quad \leqslant \gamma(s, t, h, h', \lambda)\, \|f\|_{\mu, 2}
\end{aligned} \tag{5.13}$$

where

$$\gamma^2(s, t, h, h', \lambda) = \text{area}(E(s, t, h, h')\, \Delta E(\lambda s, \lambda t, \lambda h, \lambda h')),$$
$$E(s, t, h, h') = [s, s+h] \times [t, t+h']$$

and $A \Delta B$ denote the symmetric difference of A and B.

There are two cases to consider. They are:

(i) $\lambda(s+h) \leqslant s$ or $\lambda(t+h') \leqslant t$,

(ii) $\lambda(s+h) > s$ and $\lambda(t+h') > t$.

If (i) holds then

$$\gamma^2(s, t, h, h', \lambda) = hh' + \lambda^2 hh' \tag{5.14}$$

and if (ii) holds then

$$\gamma^2(s, t, h, h', \lambda) = hh' + \lambda^2 hh' - 2(\lambda(s+h) - s)(\lambda(t+h') - t) \tag{5.15}$$

Also, if (i) holds then either $(1-\lambda)\, s \geqslant \lambda h$ or $(1-\lambda)\, t \geqslant \lambda h'$, so Eq. (5.14) implies

$$\begin{aligned}
\gamma(s, t, h, h', \lambda)/(hh')^\alpha &\leqslant (1+\lambda^2)^{1/2}\, (hh')^{1/2-\alpha} \leqslant (1+\lambda^2)^{1/2} \\
&\quad \times ((1-\lambda)/\lambda)^{1/2-\alpha} \leqslant 4(1-\lambda)^{1/2-\alpha}
\end{aligned} \tag{5.16}$$

since $1/2 < \lambda < 1$, $0 < h$, $h' < 1$. On the other hand, if (ii) holds, then drawing the appropriate picture we see

$$\begin{aligned}
\gamma^2(s, t, h, h', \lambda) &\leqslant h(1-\lambda)(t+h') + h'(1-\lambda)(s+h) \\
&\quad + h's(1-\lambda) + ht(1-\lambda) \\
&\leqslant 2h(1-\lambda)(t+h') + 2h'(1-\lambda)(s+h)
\end{aligned}$$

Now when (ii) holds,

$$\begin{aligned}
&\gamma(s, t, h, h', \lambda)/(hh')^{\alpha}\\
&\quad\leqslant 2(h(1-\lambda)(t+h'))^{1/2}/(hh')^{\alpha}+2(h'(1-\lambda)(s+h))^{1/2}/(hh')^{\alpha}\\
&\quad= 2h^{1/2-\alpha}((1-\lambda)(t+h')/h')^{\alpha}\,((1-\lambda)(t+h'))^{1/2-\alpha}\\
&\qquad+2(h')^{1/2-\alpha}\,((1-\lambda)(s+h)/h)^{\alpha}\,((1-\lambda)(s+h))^{1/2-\alpha}\\
&\quad\leqslant 2((1-\lambda)(t+h'))^{1/2-\alpha}+2((1-\lambda)(s+h))^{1/2-\alpha}\\
&\quad\leqslant 4(1-\lambda)^{1/2-\alpha}
\end{aligned}$$

since $(1-\lambda)(t+h')=(t+h')-\lambda(t+h')\leqslant h'$, $(1-\lambda)(l+h)\leqslant h$ and $0<t+h'\leqslant 1$, $0<s+h\leqslant 1$. Thus the lemma is proved.

The analogue of Lemma 4.3 in the two parameter setting is the following lemma.

Lemma 5.2. Let m, n, r be nonnegative integers with $m\leqslant n\leqslant r$ and $b_n=n(L_2 n)^{1/2}$ for $n\geqslant 1$. Then for all $f\in H_{\mu,2}$ and $1/2<p<1$, $0<p'<1/2$

$$\begin{aligned}
&(L_2 n)^p\,(L_3 n)^{-p'}\,q_{\alpha}(W(n(\cdot,\cdot))/b_n-f)\\
&\quad\geqslant\left(\frac{n}{m}\right)^{2\alpha}\cdot\frac{m}{r}\cdot\left(\frac{L_2 r}{L_2 m}\right)^{p-1/2}\left(\frac{L_3 r}{L_3 m}\right)^{-p'}\\
&\qquad\times(L_2 m)^p\,(L_3 m)^{-p'}\,q_{\alpha}(W(m(\cdot,\cdot))/b_m-f)\\
&\qquad-4\left(\frac{r}{m}\right)^{2\alpha}(L_2 r)^p\,(L_3 r)^{-p'}\left|1-\frac{m}{r}\right|^{1/2-\alpha}\|f\|_{\mu,2}\\
&\qquad-\left(\frac{r}{m}\right)^{\alpha}(L_2 r)^p\,(L_3 r)^{-p'}\left(1-\frac{b_m}{b_r}\right)q_{\alpha}(f)
\end{aligned}\tag{5.17}$$

Proof. Since $W(n(\frac{m}{n}(\cdot,\cdot))=W(m(\cdot,\cdot))$ we have

$$\begin{aligned}
&(L_2 n)^p\,(L_3 n)^{-p'}\,q_{\alpha}(W(n(\cdot,\,\cdot)/b_n-f)\\
&\quad\geqslant\left(\frac{n}{m}\right)^{2\alpha}\frac{(L_2 n)^p}{b_n}\,(L_3 n)^{-p'}\,q_{\alpha}\!\left(W(m(\cdot,\cdot))-b_n f\left(\frac{m}{n}(\cdot,\cdot)\right)\right)\\
&\quad\geqslant\left(\frac{n}{m}\right)^{2\alpha}\frac{(L_2 r)^p\,(L_3 r)^{-p'}}{b_r}\,q_{\alpha}(W(m(\cdot,\cdot))-b_n h)
\end{aligned}\tag{5.18}$$

where $h(\cdot,\cdot)=f(\frac{m}{n}(\cdot,\cdot))$. Now

$$q_\alpha(W(m(\cdot,\cdot))-b_n h(\cdot,\cdot))\geqslant q_\alpha(W(m(\cdot,\cdot))-b_m f)-b_r q_\alpha(f-h) - (b_r-b_m)\,q_\alpha(f) \tag{5.19}$$

and since $f\in H_{\mu,2}$ Lemma 5.1 implies

$$q_\alpha(f-h)\leqslant 4\,|1-m/r|^{1/2-\alpha}\,\|f\|_{\mu,2} \tag{5.20}$$

Combining Eqs. (5.18)–(5.20) we get Eq. (5.17), so the lemma is proved.

Proof of Theorem 5.1. The proof of Theorem 5.1 now follows as that for Theorem 4.1. The two things that perhaps needs some mention are the analogue of the large deviation result used for Eq. (4.44) and also Eq. (4.17) in the setting of the Brownian sheet. However, neither present a problem since $(H_{q_\alpha,0}, q_\alpha(\cdot))$ is a separable Banach space and the Brownian sheet induces a centered Gaussian measure μ_2 on the Borel subsets of this space. Furthermore, the Hilbert space generating μ_2 is $H_{\mu,2}$, and hence the large deviation result required is a special application of the large deviation theorem for Gaussian measure on a separable Banach space. The analogue of Eq. (4.17) again follows from Goodman and Kuelbs[10] [Theorem 2.1], applied as indicated following Eq. (4.17).

NOTE ADDED IN PROOF

In view of an improvement of the original results in Kuelbs et al.,[11] the isomorphisms used in the proofs of Propositions 5 and 6 are no longer needed. In particular, Theorem 1 and Propositions 1 and 2 in Kuelbs et al.[11] give (4.9), (4.10), (5.8), and (5.9) immediately.

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