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Capture time of Brownian pursuits

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Abstract. Let $B_0, B_1, \cdots, B_n$ be independent standard Brownian motions, starting at 0. We investigate the tail of the capture time

$$\tau_n = \inf\{t > 0 : B_i(t) - b_i = B_0(t) \text{ for some } 1 \le i \le n\}$$

where $0 < b_i \le 1, 1 \le i \le n$. In particular, we have $\mathbb{E}\,\tau_3 = \infty$ and $\mathbb{E}\,\tau_5 < \infty$. Various generalizations are also studied.

1. Introduction

Let $B_0, B_1, \cdots, B_n$ be independent standard Brownian motions, starting at 0. Define the stopping time

$$\tau_n = \inf\{t > 0 : B_i(t) - b_i = B_0(t) \text{ for some } 1 \le i \le n\},$$

where $0 < b_i \le 1, 1 \le i \le n$. The τ_n can be viewed as a capture time in a random pursuit setting. Assume that a Brownian prisoner escapes, running along the path of B_0. In his pursuit, there are n independent Brownian policemen. These policemen run along the paths of $B_1, \cdots, B_n$, respectively. At the outset, the prisoner is ahead of the policemen by some fixed distances b_i, $1 \le i \le n$. Then, τ_n represents the capture time when the fastest of the policemen catches the prisoner.

In an elegant paper on coupling various stochastic processes, Bramson and Griffeath (1991) considered the analogous stopping time $\tilde{\tau}_n$ for continuous time random walks. It is very likely that the kind of tail estimates which we derive here for τ_n are the same for $\tilde{\tau}_n$. However, for our purposes Brownian motions are easier to work with, so that we will stick with the setup described above.

Bramson and Griffeath raised the question: For which n is $\mathbb{E}\,\tau_n < \infty$. A more animated interpretation is "*How many Brownian policemen does it take to arrest a Brownian prisoner?*" They showed for continuous time random walks that $\mathbb{E}\,\tilde{\tau}_n = \infty$ for $n = 2$ or 3, and their computer simulations indicated that $\mathbb{E}\,\tilde{\tau}_n < \infty$ for $n \ge 4$.

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Of course τ_n equals the first exit time by the $(n+1)$-dimensional Brownian motion $(B_0(t), \cdots, B_n(t))$ from the "wedge"

$$\{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i - x_0 < 0, 1 \le i \le n\} \tag{1.1}$$

starting at $(0, -b_1, \cdots, -b_n)$. DeBlassie (1987) (see also DeBlassie (1988) and Bañuelos and Smits (1997)) has shown that

$$\mathbb{P}\{\tau_n > t\} \sim c(b)t^{-\gamma_n}, \quad \text{as} \quad t \to \infty, \tag{1.2}$$

when $b_0 = 0 < b_i, b = (b_0, -b_1, \cdots, -b_n)$, where γ_n is determined by the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on a subset of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Exact formula for γ_n is given in next section. However, as Bramson and Griffeath point out, it seems very difficult to find γ_n explicitly by this direct approach and it even seems difficult to show $\gamma_n > 1$ for n large by this method. Of course, $\gamma_1 = 1/2$ by the reflection principle, and the analysis indicated in Bramson and Griffeath (1991) shows that $\gamma_2 = 3/4$, $\gamma_3 < 1$. Further, their simulation suggested that $\gamma_3 \approx 0.91$, $\gamma_4 \approx 1.032$ and $\gamma_{10} \approx 1.4$.

Using closely related independent stationary Ornstein-Uhlenbeck processes and the theory of Large Deviations, Kesten (1992) proves that

$$c_1 \log n \le \gamma_n \le c_2 \log n$$

for large enough n and hence $\mathbb{E}\,\tau_n < \infty$ for large enough n. Here and throughout this paper, we use letter c and its modifications c', c_1 etc for various positive constants which may be different from line to line.

In this paper, we prove in particular the following

Theorem 1.1. *Let γ_n be given in (1.2). Then*

$$\gamma_2 = 3/4, \quad \gamma_3 < 1 \quad \textit{and} \quad \gamma_5 > 1.$$

Thus $\mathbb{E}\,\tau_3 = \infty$ and $\mathbb{E}\,\tau_5 < \infty$.

For fixed n, our approach is based on some distribution identities which reduce the first exit problem by $n+1$ independent Brownian motions from the "wedge" region (1.1) into a first exit problem by n independent Brownian motions from a "nicer" region. Then we can apply the well known results of DeBlassie (1987). The difference is that γ_n in (1.2) is now determined by the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on the "nicer" subset of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. In essence, we reduce the dimension by one and the new exit region is closer to a right circular cone so that we can estimate γ_3 by an inscribed right circular cone and γ_5 by the celebrated Faber-Krahn isoperimetric inequality on spheres. Note that this idea was used in an implicit way to show $\gamma_2 = 3/4$ and $\gamma_3 < 1$ in Bramson and Griffeath (1991). So far we can not push this approach to settle the conjecture $\gamma_4 > 1$, or equivalently $\mathbb{E}\,\tau_4 < \infty$ of Bramson and Griffeath. However, as mentioned above, our method represents the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on a suitable subset of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ by one in lower dimension. The full strength of this approach and its various other applications to principal eigenvalues and harmonic functions will be given in Li and Shao (2000). Some close related applications to various capture times are given in sections 4 and 5.

Also worth pointing out is a general framework for this type of problems, and what leads us to this work from the point of view of the theory of Gaussian processes. Let us assume $b_i = 1$, $1 \le i \le n$, for simplicity. First note that estimating the tail of τ_n, i.e. $\mathbb{P}(\tau_n > t)$ as $t \to \infty$, is the same as estimating the lower level boundary crossing for the Gaussian process $X(k,s) = B_k(s) - B_0(s)$ indexed by $(k,s) \in \{1,\cdots,n\} \times [0,1]$, i.e. $\mathbb{P}\left(\max_{1\le k\le n}\sup_{0\le s\le 1}(B_k(s) - B_0(s)) \le \varepsilon\right)$ as $\varepsilon \to 0$. In fact, for any $t > 0$, by Brownian scaling,

$$\begin{aligned}\mathbb{P}(\tau_n > t) &= \mathbb{P}\left(\max_{1\le k\le n}\sup_{0\le s\le t}(B_k(s) - B_0(s)) < 1\right)\\ &= \mathbb{P}\left(\max_{1\le k\le n}\sup_{0\le s\le 1}(B_k(s) - B_0(s)) < t^{-1/2}\right).\end{aligned}$$

Thus the problem can be viewed as a lower level boundary crossing problem for a real valued Gaussian random process X_t indexed by $t \in T$ with mean zero, which can be formulated as the asymptotic behavior of the probability

$$\mathbb{P}\left(\sup_{t\in T}(X_t - X_{t_0}) \le \varepsilon\right), \quad \text{as} \quad \varepsilon \to 0, \tag{1.3}$$

with $t_0 \in T$ fixed. This is different from the small ball problem under the sup-norm, which considers the absolute value of the supremum of a Gaussian process. That class of problems has been studied very recently in Bass, Eisenbaum and Shi (2000) for two sided fractional Brownian motion and generalized in Marcus (2000) to a larger class of Gaussian processes with stationary increments. Their upper bound estimate involves a clever application of Slepian's lemma, which reduces the problem to the consideration of the probability that planar Brownian motion spends a unit of time in a certain cone. In Csaki, Khoshnevisan and Shi (2000), where we learned the pursuit problem, a variation of the random pursuit problem for Brownian particles is used in their upper estimate of $\log \mathbb{P}\left(\sup_{0\le s,t\le 1} W(s,t) \le \varepsilon\right)$ where $W(s,t)$ is the standard two dimensional Brownian sheet. More precisely, they used an upper bound estimate for

$$\mathbb{P}\left(\max_{1\le k\le n}\sup_{0\le s\le 1}(B_k(s) - \delta B_0(s)) < \varepsilon\right)$$

with $\delta > 0$, which is obtained along the approach of Kesten (1992). Very recently, Li and Shao (1999) provide general upper and lower estimates for the probability in (1.3). In particular, sharp rates of $(\log \varepsilon^{-1})^d$ are obtained at the logarithmic level for d dimensional Brownian sheets. For the fractional Brownian motion on [0, 1], the exact power of a polynomial rate is unknown and seems challenging to find. See Sinai (1997), Li and Shao (1999).

Our way of attacking such problems via reduction of dimension also allow us to find certain exit probabilities. In our last section, we consider two policemen coming from both sides of a prisoner. In this case, we can compute more things and compare with the well-known exit time of a Brownian motion $B_0(t)$ from the interval $[a, b]$ with $a < 0 < b$. To make everything more precise, let

$$\tau_{ab} = \inf\{t > 0 : \ B_0(t) - B_1(t) = a \text{ or } B_0(t) - B_2(t) = b\}$$

for $a < 0 < b$.

Theorem 1.2. *We have*

$$\mathbb{E}\,\tau_{ab} = |ab| = -ab, \quad \mathbb{P}\,(\tau_{ab} > t) \sim c_{a,b} t^{-3/2} \tag{1.4}$$

as $t \to \infty$ and

$$\mathbb{P}\,(B_0(\tau_{ab}) - B_1(\tau_{ab}) = a) = \frac{1}{2} + \frac{3}{\pi} \arctan \frac{a+b}{\sqrt{3}(b-a)} \tag{1.5}$$

$$\mathbb{P}\,(B_0(\tau_{ab}) - B_2(\tau_{ab}) = b) = \frac{1}{2} - \frac{3}{\pi} \arctan \frac{a+b}{\sqrt{3}(b-a)} \tag{1.6}$$

Several remarks are in order here. First, the result $\mathbb{E}\,\tau_{ab} = -ab$ was implicitly mentioned in Bramson and Griffeath (1991). The expected capture time of various discrete models are also computed explicitly there. Second, the above results should be compared with the well-known exit time

$$\sigma_{ab} = \inf\{t > 0 : \ B_0(t) = a \text{ or } B_0(t) = b\},$$

which can be viewed as capture time by deterministic barrier lines. We note that $\mathbb{E}\,\tau_{ab} = \mathbb{E}\,\sigma_{ab} = |ab|$ but $\mathbb{E}\,\tau_{ab}^2 = \infty$, $\mathbb{E}\,\sigma_{ab}^2 < \infty$. Furthermore, from the inequality $x > \sqrt{3}\tan(\pi x/6)$ for $1 > x > 0$, we have

$$\mathbb{P}\,(B_0(\tau_{ab}) - B_1(\tau_{ab}) = a) > \mathbb{P}\,(B_0(\sigma_{ab}) = a) = b/(b-a) \tag{1.7}$$

for $b > |a|$ and the inequality is reversed if $b < |a|$. Hence it is more (less) likely to be captured by the nearer (further) one for random pursuit than the deterministic pursuit or exit. Third, the related problems for random walks are slightly different and seem hard to find exact formulas. Fourth and most importantly, the method here may allow us to connect and find harmonic functions on various wedge type regions in higher dimensions. Finally, by following the proof of Theorem 1.2, we have for $b > a > 0$,

$$\mathbb{P}\,(B_1(\tau_3(a,b)) - B_0(\tau_3(a,b)) = a) = \frac{1}{2} + \frac{3}{2\pi} \arctan \frac{\sqrt{3}(b-a)}{a+b},$$

$$\mathbb{P}\,(B_2(\tau_3(a,b)) - B_0(\tau_3(a,b)) = b) = \frac{1}{2} - \frac{3}{2\pi} \arctan \frac{\sqrt{3}(b-a)}{a+b}$$

where $\mathbb{E}\,\tau_3(a,b) = \infty$ and

$$\tau_3(a,b) = \inf\{t > 0 : \ B_1(t) - B_0(t) = a \text{ or } B_1(t) - B_0(t) = b\}.$$

The paper is organized as follows. In section 2, we summarize needed works on cones, based on Bañuelos and Smits (1997), DeBlassie (1987), Burkholder (1977) and Spitzer (1958) since we need both the general formulations and detailed computations. In section 3, we state and prove our results based on our key distribution identities to make the reduction. It allows us to conclude $\mathbb{E}\,\tau_3 = \infty$ and $\mathbb{E}\,\tau_5 < \infty$. In section 4 we present various results on different capture times for pursuit problems of m prisoners and n policemen, $m = 1, 2$ and $n = 1, 2, 3$. They can all be viewed as coupling times. The main approach is still based on distribution identities of different kinds. Several conjectures are proposed at the end of the section. In particular, we make the conjecture that the probability of capturing ALL n prisoners by n policemen chasing from one side, before a given time, is a strictly decreasing function of n. Finally, we prove Theorem 1.2 in section 5. As mentioned before, the full extent of our approach and its various other applications will be given in an upcoming paper.

2. First exit time from a generalized cone

To state precisely the results on the first exit time from a generalized cone as formulated in Bañuelos and Smits (1997), we need some notations. Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. If D is a proper open connected subset of $\mathbb{S}^{n-1}$, the generalized cone C generated by D is the set of all rays emanating from the origin 0 and passing through D. Assume that D is regular for the Dirichlet problem with respect to $L_{\mathbb{S}^{n-1}}$, the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. With this assumption we have a complete set of orthonormal eigenfunctions m_j with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 < \cdots$ satisfying

$$\begin{cases} L_{\mathbb{S}^{n-1}} m_j = -\lambda_j m_j & \text{on } D \\ m_j = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

Set

$$\alpha_j = \sqrt{\lambda_j + \left(\frac{n}{2} - 1\right)^2}, \quad a_j = \alpha_j - \left(\frac{n}{2} - 1\right) > 0$$

and

$$H_j = \frac{\Gamma((a_j + n)/2)}{\Gamma(a_j + n/2)} \int_D m_j(\theta) d\sigma(\theta),$$

where $\sigma = \sigma_{n-1}$ is the normalized spherical measure on $\mathbb{S}^{n-1}$. The confluent hypergeometric function is, with $b > 0$,

$$F_1(a, b, z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots .$$

Let $\{Z_t, t \ge 0\}$ be the n–dimensional Brownian motion, denote by $\mathbb{E}_x$ and $\mathbb{P}_x$ the expectation and probability associated with this motion starting at x, and denote by $\tau_C = \inf\{t > 0 : Z_t \notin C\}$ its first exit time from C.

Theorem A. Let C be a generalized cone in $\mathbb{R}^n$. Then

$$\mathbb{P}_x\{\tau_C > t\} = \sum_{j=1}^{\infty} H_j \left(\frac{|x|^2}{2t}\right)^{a_j/2} F_1\left(\frac{a_j}{2}, a_j + \frac{n}{2}, \frac{-|x|^2}{2t}\right) m_j\left(\frac{x}{|x|}\right), \quad (2.2)$$

uniformly for $(x, t) \in K \times (T, \infty)$, where $K \subset C$ is compact and $T > 0$. In particular, for each $x \in C$,

$$\mathbb{P}_x\{\tau_C > t\} \sim H_1 m_1 (x/|x|) \left(|x|^2/2\right)^{a_1/2} t^{-a_1/2} \quad (2.3)$$

as $t \to \infty$, and hence for $p > 0$

$$\mathbb{E}_x(\tau_C^p) < \infty \quad \text{if and only if } p < a_1/2. \quad (2.4)$$

The above result was first proved by DeBlassie (1987) under somewhat stronger assumptions on the cones. (2.4) also follows from Lemma 3.1 in Bass and Burdzy (1996). The special geometric structure of the cone (scale invariance) is essential for these results. When C is the right circular cone of angle $0 < \theta < \pi$ given by $\Gamma = \Gamma_\theta = \{x \in \mathbb{R}^n : \phi(x) < \theta\}$, where $\phi(x)$ is the angle between $x \in \mathbb{R}^n \setminus \{0\}$ and the point $(1, 0, \cdots, 0) \in \mathbb{R}^n$, we have the following more explicit result given in DeBlassie (1987), based on Burkholder (1977).

Theorem B. Let $\Gamma = \Gamma_\theta = \{x \in \mathbb{R}^n : \phi(x) < \theta\}$. Then for each $x \in \Gamma$,

$$\mathbb{P}_x\{\tau_\Gamma > t\} \sim c(x, \theta, n) t^{-p(\theta,n)/2} \quad (2.5)$$

as $t \to \infty$, and hence for $r > 0$

$$\mathbb{E}_x(\tau_\Gamma^r) < \infty \quad \text{if and only if } r < p(\theta, n)/2, \quad (2.6)$$

where the mapping $\theta \to p(\theta, n)$ is the inverse of $p \to \theta(p, n)$ and $\theta(p, n)$ is the smallest positive zero θ of

$$F(-p, p + n - 2; (n - 1)/2, (1 - \cos\theta)/2), \quad (2.7)$$

where $F(\alpha, \beta; y, z)$ is the hypergeometric function given by

$$F(\alpha, \beta; y, z) = 1 + \frac{\alpha\beta}{y}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{y(y+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{y(y+1)(y+2)}\frac{z^3}{3!} + \cdots$$

for $y > 0$, $|z| < 1$.

For $n = 2$, $p(\theta, 2) = \pi/(2\theta)$. For $n \geq 3$, $p \to \theta(p, n)$ is continuous and strictly decreasing from $(0, \infty)$ onto $(0, \pi)$ with $\theta(1, n) = \pi/2$, see Burkholder (1977), pp. 192–193. Thus $\theta \to p(\theta, n)$ is continuous and strictly decreasing from $(0, \pi)$ onto $(0, \infty)$, and $p(\pi/2, n) = 1$.

We should mention here that formulas for $\mathbb{P}_x\{\tau_\Gamma > t\}$ in $\mathbb{R}^2$ have existed for many years. Indeed, Spitzer (1958) in his study of the winding of two dimensional

Brownian motion derives an expression for $\mathbb{P}_x\{\tau_\Gamma > t\}$ from which the two dimensional case (2.5) and (2.6) follows. Furthermore, as pointed out in Bramson and Griffeath (1991), for $0 < \theta < \arctan\sqrt{n-1}$

$$\mathbb{E}_x\tau_\Gamma = \frac{x_1^2\sec^2\theta - |x|^2}{(n-1)-\tan^2\theta}, \tag{2.8}$$

where $x \in \Gamma$ and

$$\Gamma = \Gamma_\theta = \{x \in \mathbb{R}^n : \phi(x) \le \theta\} = \{x \in \mathbb{R}^n : |x| \le x_1 \sec\theta\}.$$

This can be proved easily by verifying that $u(x) = \mathbb{E}_x\tau_\Gamma$ for $x \in \mathbb{R}^n$ is the unique solution of Poisson's equation $\Delta u = -2$ in Γ vanishing at the boundary. See Dynkin and Yushkevich (1969) for similar arguments.

Next we find the *critical* angle θ_n of the largest right circular cone in $\mathbb{R}^n$ such that $\mathbb{E}\,\tau_\Gamma < \infty$ for all $0 < \theta < \theta_n$. Formula (2.8) suggests the answer

$$\theta_n = \arctan\sqrt{n-1} \quad \text{or equivalently} \quad \theta_n = \arccos(1/\sqrt{n}), \tag{2.9}$$

which is correct and can be proved in the following way. Take $p(\theta, n) = 2$ in (2.7) and observe from the definition of the hypergeometric function that

$$F(-2, n; \frac{n-1}{2}, x) = 1 - \frac{4n}{n-1}x + \frac{4n}{n-1}x^2.$$

It is easy to see that the smallest positive solution is $x = (1 - n^{-1/2})/2$. (2.9) now follows from (2.6) and (2.7).

3. Main results and proofs

For simplicity, we assume $b_i = 1$, $1 \le i \le n$ in this section. The general case can be obtained similarly and the power γ_n in (1.2) is independent of $0 < b_i \le 1$, $1 \le i \le n$, which can be seen from (2.2) and (2.3). Throughout the remainder of this paper, $\{W_k(t);\ t \ge 0\}(k = 0, 1, 2, \dots)$ denote independent Brownian motions all starting from 0.

Let $Y_i(t) = B_i(t) - B_0(t)$, $1 \le i \le n$, $0 \le t \le 1$. Then $Y_i(t)$ is a mean zero Gaussian process indexed by $T_n = \{(i, t) : 1 \le i \le n, 0 \le t \le 1\}$ and $Y_i(0) = 0$ for $1 \le i \le n$. The covariance of $Y_i(t)$ can be computed easily

$$\mathbb{E}\,Y_i(s)Y_j(t) = \begin{cases} \min(s,t) & \text{if } i \ne j,\ 1 \le i, j \le n,\ 0 \le s, t \le 1\ , \\ 2\min(s,t) & \text{if } i = j,\ 1 \le i \le n,\ 0 \le s, t \le 1 \end{cases}.$$

Our key observation is the fact that the Gaussian process $\{Y_i(t) : (i, t) \in T_n\}$ and $\{X_i(t) : (i, t) \in T_n\}$ are the same in law, where

$$\begin{aligned}
X_1(t) &= 2^{1/2}W_1(t), \\
X_2(t) &= 2^{-1/2}W_1(t) + (3/2)^{1/2}W_2(t), \\
X_3(t) &= 2^{-1/2}W_1(t) + 6^{-1/2}W_2(t) + (4/3)^{1/2}W_3(t), \\
X_4(t) &= 2^{-1/2}W_1(t) + 6^{-1/2}W_2(t) + (12)^{-1/2}W_3(t) + (5/4)^{1/2}W_4(t), \\
X_5(t) &= 2^{-1/2}W_1(t) + 6^{-1/2}W_2(t) + (12)^{-1/2}W_3(t) \\
&\quad + 20^{-1/2}W_4(t) + (6/5)^{1/2}W_5(t)
\end{aligned}$$

and in general for $i \geq 1$,

$$X_i(t) = \sum_{k=1}^{i} a_{i,k} W_k(t)$$

with $a_{i,k} = ((k+1)k)^{-1/2}$ for $1 \leq k \leq i-1$ and $i \geq 2$, and $a_{i,i} = ((i+1)/i)^{1/2}$. It is easy to check that for $i < j$

$$\mathbb{E}\, X_i(s)X_i(t) = \sum_{k=1}^{i} a_{i,k}^2 \min(s,t) = 2\min(s,t),$$
$$\mathbb{E}\, X_i(s)X_j(t) = \sum_{k=1}^{i} a_{i,k}a_{j,k} \min(s,t) = \min(s,t),$$

which agrees with covariances for the Y's.

Now by using the above distribution identities for any $n \geq 1$, we have

$$\begin{aligned}\mathbb{P}\,(\tau_n > t) &= \mathbb{P}\left(\max_{1\leq i\leq n}\sup_{0\leq s\leq t}(B_i(s) - B_0(s)) < 1\right)\\ &= \mathbb{P}\left(\max_{1\leq i\leq n}\sup_{0\leq s\leq t}\sum_{k=1}^{i} a_{i,k}W_k(s) < 1\right)\\ &= \mathbb{P}\left(\max_{1\leq i\leq n}\sup_{0\leq s\leq t}\sum_{k=1}^{i} a_{i,k}(W_k(s) - a_{n+1,k}) < 0\right)\\ &= \mathbb{P}(\tau_G > t)\end{aligned}$$

where τ_G is the first exit time of the domain

$$G = G_n = \bigcap_{i=1}^{n}\left\{x = (x_k) \in \mathbb{R}^n : \sum_{k=1}^{i} a_{i,k}x_k \leq 0\right\} \tag{3.1}$$

for the standard $\mathbb{R}^n$ valued Brownian motion starting at the point

$$V = (-a_{n+1,1}, -a_{n+1,2}, \cdots, -a_{n+1,n}). \tag{3.2}$$

Note that the domain $G = G_n$ is a generalized cone with vertex at the origin.

Theorem 3.1. *We have $\tau_n = \tau_G$ in distribution and the distribution is given in (2.2) with $C = G = G_n$. In particular, for $n \geq 5$,*

$$\frac{n}{n^2-1} < \mathbb{E}\,\tau_n < \infty.$$

Proof of Theorems 3.1 and 1.1. For $n = 2$, it follows from Theorem B and the result of Spitzer that $\gamma_2 = p(\theta, 2)/2 = \pi/(4\theta) = 3/4$, where $\theta = \pi/3$ for the region

$$G_2 = \bigcap_{i=1}^{2} \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \sum_{k=1}^{i} a_{i,k} x_k \le 0 \right\}$$
$$= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0, x_1 + \sqrt{3} x_2 \le 0 \right\}.$$

Next we need to find the largest right circular cone $C_n = \{x \in \mathbb{R}^n : \phi(x, e) \le \rho_n\}$ of angle $0 < \rho_n < \pi$ inside the domain G_n, where $\phi(x, e)$ denotes the angle between x and the point $e \in \mathbb{R}^n$ with $|e| = 1$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. These can be done easily by first finding the point $e = (e_1, \cdots, e_n)$ which satisfies the equations

$$d(e, H_i) = d(e, H_j), \quad 1 \le i < j \le n \tag{3.3}$$

where

$$H_i \ : \ \sum_{k=1}^{i} a_{i,k} x_k = 0, \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

with normal vector

$$h_i = (2^{-1/2} a_{i,1}, \cdots, 2^{-1/2} a_{i,i}, 0, \cdots 0) \in \mathbb{R}^n,$$

and

$$d(e, H_i) = \langle e, h_i \rangle = \sum_{k=1}^{i} 2^{-1/2} a_{i,k} e_k$$

is the distance from e to the hyperplane H_i with normal vector h_i. Solving (3.3) yields $e = V/|V|$, where the point V is given in (3.2) and hence the angle ρ_n of the largest right circular cone inside G_n is given by

$$\sin \rho_n = d(e, H_i) = \left(\frac{n+1}{2n} \right)^{1/2}, \quad \text{i.e.} \quad \rho_n = \arctan \sqrt{(n+1)/(n-1)}$$

In particular, ρ_n is decreasing in n and $\rho_2 = \pi/3$, $\rho_3 = \arctan \sqrt{3}$, and $\lim_{n\to\infty} \rho_n = \pi/4$. Note that $\rho_3 = \theta_3$ and hence $\gamma_3 < 1$ and $\mathbb{E}\,\tau_3 = \infty$. Furthermore, the expected exit time of the largest right circular cone $C_n = \{x \in \mathbb{R}^n : \phi(x, e) \le \rho_n\}$ (inside the domain G_n) from the point $V = (-a_{n+1,1}, -a_{n+1,2}, \cdots, -a_{n+1,n})$ is

$$\frac{x_1^2 \sec^2 \rho_n - |x|^2}{(n-1) - \tan^2 \rho_n} = \frac{n}{n^2 - 1}$$

given in (2.8), with

$$x_1^2 = |x|^2 = |V|^2 = \sum_{k=1}^{n} a_{n+1,k}^2 = 2 - a_{n+1,n+1}^2 = n/(n+1)$$

and

$$\rho_n = \arctan \sqrt{(n+1)/(n-1)}.$$

Thus we have $\mathbb{E}\,\tau_n = \mathbb{E}\,\tau_G > n/(n^2-1)$ for all n.

So for the remainder of this section, we will show that $\gamma_5 > 1$ which implies $\mathbb{E}\,\tau_5 < \infty$. Note that it can be shown that the smallest right circular cone with vertex at the origin that includes G_n is

$$\{x \in \mathbb{R}^n : \phi(x, V/|V|) \le \eta_n\}$$

of angle $\eta_n = \arccos(1/n)$, where $\phi(x, e)$ denotes the angle between x and the point $V \in \mathbb{R}^n$ given in (3.2). Here the angle $\eta_n > \theta_n$ for all n and thus the smallest right circular sur-scribed cone of G_n will never allow us to show the finiteness of $\mathbb{E}\,\tau_n$ for any n. Thus, we invoke the Faber-Krahn isoperimetric inequality

$$\lambda_1(G_n \cap \mathbb{S}^{n-1}) \ge \lambda_1(G_n^* \cap \mathbb{S}^{n-1}),$$

where G_n^* is a right circular cone with vertex at the origin such that

$$m_{n-1}(G_n \cap \mathbb{S}^{n-1}) = m_{n-1}(G_n^* \cap \mathbb{S}^{n-1})$$

(see page 87, Chavel (1984)). Recall from section 2 that

$$\gamma_n = \frac{1}{2}\left(\sqrt{\lambda_1(G_n) + \left(\frac{n}{2}-1\right)^2} - \left(\frac{n}{2}-1\right)\right) \tag{3.4}$$

and for the critical right circular cone $\Gamma_n = \{x \in \mathbb{R}^n : |x| \le x_1 \sec\theta_n\}$

$$\frac{1}{2}\left(\sqrt{\lambda_1(\Gamma_n) + \left(\frac{n}{2}-1\right)^2} - \left(\frac{n}{2}-1\right)\right) = 1. \tag{3.5}$$

Hence for $n = 5$, we only need to show

$$m_4(G_5 \cap \mathbb{S}^4) < m_4(\Gamma_5 \cap \mathbb{S}^4). \tag{3.6}$$

Note that for $n = 5$, $\theta_5 = \arccos(1/\sqrt{5})$ and

$$\begin{aligned} m_4(\Gamma_5 \cap \mathbb{S}^4) &= \int_0^{\theta_5}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} \sin^3\phi_1 \sin^2\phi_2 \sin\phi_3 d\phi_4 d\phi_3 d\phi_2 d\phi_1 \\ &= 2\pi^2(1-\cos\theta_5 - (1-\cos^3\theta_5)/3) \\ &= \frac{4\pi^2}{3}\left(1 - \frac{7}{5\sqrt{5}}\right) \\ &= 4.9203... \end{aligned} \tag{3.7}$$

To calculate $m_4(G_5 \cap \mathbb{S}^4)$, let

$$\mathscr{B}_0 = \begin{pmatrix} 2^{1/2} & 0 & 0 & 0 & 0 \\ 2^{-1/2} & (3/2)^{1/2} & 0 & 0 & 0 \\ 2^{-1/2} & 6^{-1/2} & (4/3)^{1/2} & 0 & 0 \\ 2^{-1/2} & 6^{-1/2} & (12)^{-1/2} & (5/4)^{1/2} & 0 \\ 2^{-1/2} & 6^{-1/2} & (12)^{-1/2} & (20)^{-1/2} & (6/5)^{1/2} \end{pmatrix},$$

$$\mathscr{B}_1 = \begin{pmatrix} 2^{-1/2} & (3/2)^{1/2} & 0 & 0 & 0 \\ 2^{-1/2} & 6^{-1/2} & (4/3)^{1/2} & 0 & 0 \\ 2^{-1/2} & 6^{-1/2} & (12)^{-1/2} & (5/4)^{1/2} & 0 \\ 2^{-1/2} & 6^{-1/2} & (12)^{-1/2} & (20)^{-1/2} & (6/5)^{1/2} \end{pmatrix}$$

and

$$\mathscr{B}(\varepsilon) = \begin{pmatrix} -2^{-1/2} & (3/2)^{1/2}\varepsilon_2 & 0 & 0 & 0 \\ -2^{-1/2} & 6^{-1/2}\varepsilon_2 & (4/3)^{1/2}\varepsilon_3 & 0 & 0 \\ -2^{-1/2} & 6^{-1/2}\varepsilon_2 & (12)^{-1/2}\varepsilon_3 & (5/4)^{1/2}\varepsilon_4 & 0 \\ -2^{-1/2} & 6^{-1/2}\varepsilon_2 & (12)^{-1/2}\varepsilon_3 & (20)^{-1/2}\varepsilon_4 & (6/5)^{1/2}\varepsilon_5 \end{pmatrix},$$

where $\varepsilon = (\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) \in \{\pm 1\}^4$.

For $m \geq 2$ and $1 \leq i, j, k, l \leq m$ set

$$\begin{aligned} U_{i,j,k,l} = U_{i,j,k,l}(m) = & \left[\frac{(i-1)\pi}{2m}, \frac{i\pi}{2m}\right] \times \left[\frac{(j-1)\pi}{2m}, \frac{j\pi}{2m}\right] \\ & \times \left[\frac{(k-1)\pi}{2m}, \frac{k\pi}{2m}\right] \times \left[\frac{(l-1)\pi}{2m}, \frac{l\pi}{2m}\right]. \end{aligned}$$

Write

$$\begin{aligned} G_5 \cap \mathbb{S}^4 &= \{x \in \mathbb{S}^4 : \mathscr{B}_0 x' \leq 0\} = \{\phi \in [0,\pi]^3 \times [0, 2\pi],\ \mathscr{B}_0 x' \leq 0\} \\ &= \{\phi \in [\pi/2, \pi] \times [0, \pi]^2 \times [0, 2\pi],\ \mathscr{B}_1 x' \leq 0\}, \end{aligned}$$

where $x = (x_1, x_2, ..., x_5)$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ and

$$\begin{aligned} x_1 &= \cos\phi_1, \\ x_2 &= \sin\phi_1 \cos\phi_2, \\ x_3 &= \sin\phi_1 \sin\phi_2 \cos\phi_3, \\ x_4 &= \sin\phi_1 \sin\phi_2 \sin\phi_3 \cos\phi_4, \\ x_5 &= \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4, \end{aligned}$$

by using polar coordinates. Thus, we have

$$
\begin{aligned}
& m_4(G_5 \cap \mathbb{S}^4) \\
&\quad = \int_{\{\phi \in [\pi/2,\pi]\times[0,\pi]^2\times[0,2\pi],\ \mathscr{B}_1 x' \le 0\}} \sin^3\phi_1 \sin^2\phi_2 \sin\phi_3 d\phi_4 d\phi_3 d\phi_2 d\phi_1 \\
&\quad = \sum_{\varepsilon\in\{\pm1\}^4} \int_{\{\phi\in[0,\pi/2]^4:\ \mathscr{B}(\varepsilon)x'\le 0\}} \sin^3\phi_1 \sin^2\phi_2 \sin\phi_3 d\phi_4 d\phi_3 d\phi_2 d\phi_1 \\
&\quad = \sum_{\varepsilon\in\{\pm1\}^4} \sum_{1\le i,j,k,l\le m} \int_{\{\phi\in U_{i,j,k,l}:\ \mathscr{B}(\varepsilon)x'\le 0\}} \sin^3\phi_1 \sin^2\phi_2 \sin\phi_3 d\phi_4 d\phi_3 d\phi_2 d\phi_1 \\
&\quad \le \sum_{\varepsilon\in\{\pm1\}^4} \sum_{1\le i,j,k,l\le m} 1_{\{\phi(i,j,k,l,\varepsilon)\le 0\}} \int_{\phi\in U_{i,j,k,l}} \\
&\qquad \sin^3\phi_1 \sin^2\phi_2 \sin\phi_3 d\phi_4 d\phi_3 d\phi_2 d\phi_1 \\
&\quad = \sum_{\varepsilon\in\{\pm1\}^4} \sum_{1\le i,j,k,l\le m} 1_{\{\phi(i,j,k,l,\varepsilon)\le 0\}} \\
&\quad \Big\{ \frac{\pi}{2m}\Big(\cos\frac{(k-1)\pi}{2m} - \cos\frac{k\pi}{2m}\Big) \Big(\frac{\pi}{4m} - \frac{1}{4}\Big(\sin\frac{l\pi}{m} - \sin\frac{(l-1)\pi}{m}\Big)\Big) \\
&\quad \times\Big(\cos\frac{(i-1)\pi}{2m} - \cos\frac{i\pi}{2m}\Big) - \frac{1}{3}\Big(\cos^3\frac{(i-1)\pi}{2m} - \cos^3\frac{i\pi}{2m}\Big)\Big)\Big)\Big\} \\
&\quad := K_m,
\end{aligned}
\tag{3.8}
$$

where $\phi(i,j,k,l,\varepsilon) = \mathscr{B}(\varepsilon)x(i,j,k,l,\varepsilon)'$ and $x(i,j,k,l,\varepsilon) = (x_1^*, x_2^*, ..., x_5^*)$ with

$$
\begin{aligned}
x_1^* &= \cos((i-1)\pi/(2m)), \\
x_2^* &= \sin((i-(\varepsilon_2+1)/2)\pi/(2m))\cos((j+(\varepsilon_2-1)/2)\pi/(2m)), \\
x_3^* &= \sin((i-(\varepsilon_2+1)/2)\pi/(2m))\sin((j-(\varepsilon_2+1)/2)\pi/(2m)) \\
&\quad \times\cos((k+(\varepsilon_3-1)/2)\pi/(2m)), \\
x_4^* &= \sin((i-(\varepsilon_2+1)/2)\pi/(2m))\sin((j-(\varepsilon_2+1)/2)\pi/(2m)) \\
&\quad \times\sin((k-(\varepsilon_3+1)/2)\pi/(2m))\cos((l+(\varepsilon_4-1)/2)\pi/(2m)), \\
x_5^* &= \sin((i-(\varepsilon_2+1)/2)\pi/(2m))\sin((j-(\varepsilon_2+1)/2)\pi/(2m)) \\
&\quad \times\sin((k-(\varepsilon_3+1)/2)\pi/(2m))\sin((l-(\varepsilon_4+1)/2)\pi/(2m)).
\end{aligned}
$$

In particular, when $m = 50$, a direct calculation gives

$$K_{50} = 4.7078... \tag{3.9}$$

This proves (3.6), by (3.7) and (3.9).

Several remarks are in order here. First, the argument we presented here to show $\gamma_5 > 1$ does not work for the conjecture $\gamma_4 > 1$ since in that case, we have $m_3(G_4 \cap \mathbb{S}^3) > m_3(\Gamma_4 \cap \mathbb{S}^3)$ which goes the wrong way. Second, it may seem surprising in the case $n = 5$ that we have a relatively big margin when comparing (3.9) with (3.7). Afterall, both simulation results of Bramson and Griffeath (1991) and theoritical result of Kesten (1992) suggest that γ_4 and γ_5 are very close. This appearent contradiction is due to the way γ_n depend on $\lambda_1(G_n)$ in the formula (3.4). In fact, we see from (3.5) that $\lambda_n(\Gamma_n) = 2n$. Third, as we checked carefully, the isoperimetic comparision does not work for the variation representation of γ_n in terms of the wedge region (1.1) in $\mathbb{R}^{n+1}$ for $n = 4$. This is mainly due to the shape of the wedge region which is far away from a right circular cone. Finally, we mention that the estimate $\mathbb{E}\,\tau_n > n/(n^2-1)$ is only good for $n \geq 4$ small. In fact, it is not hard to show that $\mathbb{E}\,\tau_n \geq c(\log n)^{-1/2}$ for some $c > 0$.

4. *m* prisoners and *n* policemen

Let B_{-i}, $0 \leq i \leq m-1$ and B_j, $1 \leq j \leq n$ be independent Brownian motions, starting at 0. If we think of B_{-i}, $0 \leq i \leq m-1$, as prisoners and B_j, $1 \leq j \leq n$, as policemen, we can then define various capture times or coupling times in applications. For simplicity, we assume that all prisoners start one unit ahead of the policemen. Note again that different starting positions do not change the decay rate of the tails of capture times.

Define the first capture time of a prisoner by

$$\tau_{1,m,n} = \inf\{t > 0 : \max_{1\leq j\leq n} B_j(t) = \min_{0\leq i\leq m-1} B_{-i}(t) + 1\} \tag{4.1}$$

and the overall capture time of all prisoners by

$$\tau_{m,m,n} = \inf\{t > 0 : \max_{1\leq j\leq n} B_j(t) = \max_{0\leq i\leq m-1} B_{-i}(t) + 1\}. \tag{4.2}$$

Then we have

$$\mathbb{P}\left(\tau_{1,m,n} > t\right) = \mathbb{P}\left(\max_{1\leq j\leq n}\sup_{0\leq s\leq t}\max_{0\leq i\leq m-1}(B_j(s) - B_{-i}(s)) < 1\right) \tag{4.3}$$

and

$$\mathbb{P}\left(\tau_{m,m,n} > t\right) = \mathbb{P}\left(\max_{1\leq j\leq n}\sup_{0\leq s\leq t}\min_{0\leq i\leq m-1}(B_j(s) - B_{-i}(s)) < 1\right). \tag{4.4}$$

Obviously we can also define and study the capture time of exactly i prisoners. But as we have seen from the previous section, even the tail behavior of $\tau_n = \tau_{1,1,n}$ for $n \geq 3$ is hard to compute. So we only deal with several interesting cases. However, our approach of reducing dimension can be used in more general cases. Further results will be given in a separate paper.

Let us first observe that by symmetry $\tau_{1,m,1} = \tau_{1,1,m} = \tau_m$ in distribution. So we start with the capture time $\tau_{2,2,1}$ which can also be viewed as the coupling time of one particle with the other two. Further, throughout this section, we set

$$X_{ij}(t) = B_j(t) - B_{-i}(t), \quad i \geq 0, j \geq 1, t \geq 0.$$

Theorem 4.1. *We have*

$$\mathbb{P}\left(\tau_{2,2,1} > t\right) \sim ct^{-3/8} \quad as \quad t \to \infty. \tag{4.5}$$

Proof. Note that we can represent $X_{i1}(t)$, $i = 0, 1$, jointly by

$$X_{01}(t) = \sqrt{2}W_1(t),$$
$$X_{11}(t) = 2^{-1/2}W_1(t) + (3/2)^{1/2}W_2(t).$$

And thus

$$\begin{aligned}
&= \mathbb{P}\left(\tau_{2,2,1} > t\right) \\
&= \mathbb{P}\left(\sup_{0\le s\le t}\min_{0\le i\le 1} X_{ij}(s) < 1\right) \\
&= \mathbb{P}\left(\sup_{0\le s\le t}\min\left(\sqrt{2}W_1(s), 2^{-1/2}W_1(s) + (3/2)^{1/2}W_2(s)\right) < 1\right) \\
&= \left(\sup_{0\le s\le t}\min\left(W_1(s) - 2^{-1/2}, 2^{-1}(W_1(s) - 2^{-1/2})\right.\right. \\
&\qquad \left.\left. + (3/4)^{1/2}(W_2(s) - 6^{-1/2})\right) < 0\right) \\
&= \mathbb{P}\left(\tau_G > t\right)
\end{aligned}$$

where τ_G is the first exit time of the domain

$$G = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_1 \le 0, \text{ or}, \ 2^{-1}x_1 + (3/4)^{1/2}x_2 \le 0\} \tag{4.6}$$

for the standard $\mathbb{R}^2$ valued Brownian motion starting at the point $(-2^{-1/2}, -6^{-1/2})$. The domain G in (4.6) is a cone with $2\theta = 2\pi - 2\pi/3 = 4\pi/3$ and hence $p(\theta, n)/2 = 3/8$ for $n = 2$ and $\theta = 2\pi/3$. Our result follows from the work of Spitzer (1958) as mentioned in section 2.

We next turn to the capture times $\tau_{1,2,2}$, $\tau_{2,2,2}$ and $\tau_{1,2,3}$.

Theorem 4.2. *We have* $\mathbb{E}\,\tau_{1,2,3} \le \mathbb{E}\,\tau_{1,2,2} \le \mathbb{E}\,\tau_4$ *and* $\mathbb{E}\,\tau_{1,2,3} \le \mathbb{E}\,\tau_6 \le \mathbb{E}\,\tau_5 < \infty$. *Furthermore,* $\mathbb{P}(\tau_{1,2,2} > t) \le \mathbb{P}(\tau_4 > t)$, $\mathbb{P}(\tau_{1,2,2} > t) \le ct^{-1}$ *and* $\mathbb{P}(\tau_{1,2,3} > t) \le \mathbb{P}(\tau_6 > t)$.

Proof. Let us first work on the case $m = n = 2$. Note that we can represent $X_{ij}(t)$, $i = 0, 1$, $j = 1, 2$, jointly by

$$X_{01}(t) = \sqrt{2}W_1,$$
$$X_{11}(t) = 2^{-1/2}W_1 + 2^{-1/2}W_2 + W_3,$$
$$X_{02}(t) = 2^{-1/2}W_1 + 2^{-1/2}W_2 - W_3,$$
$$X_{12}(t) = \sqrt{2}W_2.$$

And thus

$$\begin{aligned}
\mathbb{P}\left(\tau_{1,2,2} > t\right) &= \mathbb{P}\left(\sup_{0\le s\le t}\max_{0\le i\le 1}\max_{1\le j\le 2} X_{ij}(s) < 1\right)\\
&\le \mathbb{P}\left(\sup_{0\le s\le t}\max\left(X_{01}(s), X_{12}(s)\right) < 1\right)\\
&= \mathbb{P}\left(\sup_{0\le s\le t}\max\left(\sqrt{2}W_1(s), \sqrt{2}W_2(s)\right) < 1\right)\\
&= \mathbb{P}^2\left(\sup_{0\le s\le t} W_1(s) < 2^{-1/2}\right)\\
&= \mathbb{P}^2\left(|W_1(t)| < 2^{-1/2}\right)\\
&\le ct^{-1}.
\end{aligned}$$

Next by comparing the covariance matrix of $X_{ij}(t)$, $i = 0, 1$, $j = 1, 2$ with the one for $Y_i(t)$, $1 \le i \le 4$, given at the begining of section 3 for the case $n = 4$ and $m = 1$, we have by applying the Slepian lemma

$$\mathbb{P}(\tau_{1,2,2} > t) \le \mathbb{P}(\tau_4 > t)$$

for all $t \ge 0$ and hence $\mathbb{E}\,\tau_{1,2,2} \le \mathbb{E}\,\tau_4$.

Finally, we can work on the case of $m = 2$ and $n = 3$. Here we present two distinct but related arguments, in the order in which we found them. We first outline the proof that $\mathbb{E}\,\tau_{1,2,3} < \infty$ by the Faber-Krahn isoperimetric inequality similar to what we did for τ_5 in section 3. Note that we can represent $X_{ij}(t)$, $i = 0, 1$, $j = 1, 2, 3$, jointly by

$$\begin{aligned}
X_{01}(t) &= \sqrt{2}W_1\\
X_{11}(t) &= 2^{-1/2}W_1 + 2^{-1/2}W_2 + (2^{-1/2}W_3 + 2^{-1/2}W_4)\\
X_{02}(t) &= 2^{-1/2}W_1 + 2^{-1/2}W_2 - (2^{-1/2}W_3 + 2^{-1/2}W_4)\\
X_{12}(t) &= \sqrt{2}W_2\\
X_{03}(t) &= 2^{-1/2}W_1 + 4^{-1}(\sqrt{10} - \sqrt{2})W_3 - 4^{-1}(\sqrt{10} + \sqrt{2})W_4\\
X_{13}(t) &= 2^{-1/2}W_2 + 4^{-1}(\sqrt{10} + \sqrt{2})W_3 - 4^{-1}(\sqrt{10} - \sqrt{2})W_4
\end{aligned}$$

And thus, as before,

$$\begin{aligned}
\mathbb{P}\left(\tau_{1,2,3} > t\right) &= \mathbb{P}\left(\sup_{0\le s\le t}\max_{0\le i\le 1}\max_{1\le j\le 3} X_{ij}(s) < 1\right)\\
&= \mathbb{P}\left(\tau_G > t\right)
\end{aligned}$$

where τ_G is the first exit time of the domain

$$\begin{aligned}
G = G_{1,2,3} = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \quad & x_1 \le 0,\ \ x_2 \le 0\ \ x_1 + x_2 + x_3 + x_4 \le 0,\\
& x_1 + x_2 - x_3 - x_4 \le 0,\ \ x_1 + 2^{-1}(\sqrt{5} - 1)x_3 - 2^{-1}(\sqrt{5} + 1)x_4 \le 0,\\
& x_2 + 2^{-1}(\sqrt{5} + 1)x_3 - 2^{-1}(\sqrt{5} - 1)x_4 \le 0\}
\end{aligned}$$

for the standard $\mathbb{R}^4$ valued Brownian motion starting at the point $(-2^{-1/2}, -2^{-1/2}, -10^{-1/2}, 10^{-1/2})$. A straightforward calculation shows that

$$m_3(G_{1,2,3} \cap \mathbb{S}^3) < m_3(\Gamma_4 \cap \mathbb{S}^3) = \int_0^{\pi/3} \int_0^{\pi} \int_0^{2\pi} \sin^2 \phi_1 \sin \phi_2 d\phi_3 d\phi_2 d\phi_1 = \frac{2}{3}\pi^2 - \frac{\sqrt{3}}{2}\pi$$

which implies $\mathbb{E}\,\tau_{1,2,3} < \infty$.

Now we show that

$$\mathbb{P}\left(\tau_{1,2,3} > t\right) \le \mathbb{P}\left(\tau_6 > t\right) \tag{4.7}$$

and hence $\mathbb{E}\,\tau_{1,2,3} \le \mathbb{E}\,\tau_6 \le \mathbb{E}\,\tau_5 < \infty$ by Theorem 1.1. Rewrite

$$\mathbb{P}\left(\tau_{1,2,3} > t\right) = \mathbb{P}\left(\sup_{0 \le s \le t} \max_{(i,j) \in H} X_{ij}(s) < 1\right)$$

where $H = \{(0,1), (1,1), (0,2), (1,2), (0,3), (1,3)\}$. Comparing the covariance matrices of $\{X_{ij}, (i,j) \in H\}$ and $\{X_i, 1 \le i \le 6\}$ given in section 3 and applying Slepian lemma, (4.7) follows.

Next we mention two conjectures which are of some interest and may not be very hard, in particular the first one below. The method of solving them should be useful for the conjecture $\mathbb{E}\,\tau_4 < \infty$.

Conjecture 1. *A weaker conjecture than $\mathbb{E}\,\tau_4 < \infty$ is $\mathbb{E}\,\tau_{1,2,2} < \infty$. The estimate $\mathbb{P}\left(\tau_{1,2,2} > t\right) \le ct^{-1}$ almost worked.*

Conjecture 2. *Let*

$$\mathbb{P}\left(\tau_{n,n,n} > t\right) \sim ct^{-\mu_n} \quad \textit{as} \quad t \to \infty.$$

Then $0 < \mu_{n+1} < \mu_n < \cdots < \mu_2 < \mu_1 = 1/2$ and $\lim_{n\to\infty} \mu_n = 0$. Furthermore, $\mathbb{P}\left(\tau_{n,n,n} > t\right)$ is a strictly increasing function of n.

This seems hard to prove partially due to the mixture of max *and* min. *A related problem for Brownian sheet is studied in Csaki, Khoshnevisan and Shi (1999a) with an unknown rate. Note further that*

$$\begin{aligned}
\mathbb{P}\left(\tau_{2,2,2} > t\right) &= \mathbb{P}\left(\max_{1 \le j \le 2} \min_{0 \le i \le 1} (B_j(s) - B_{-i}(s)) < 1, \ \textit{for } 0 \le s \le t\right) \\
&= \mathbb{P}\left(\max_{1 \le j \le 2} \min_{0 \le i \le 1} X_{ij}(s) < 1, \ \textit{for } 0 \le s \le t\right) \\
&= \mathbb{P}\left(\tau_G > t\right)
\end{aligned}$$

where τ_G is the first exit time of the domain

$$\begin{aligned}
G = G_{2,2,2} = &\{x_1 \le 0, \ x_2 \le 0\} \cup \{x_1 + x_2 + \sqrt{2}x_3 \le 0, \ x_1 + x_2 - \sqrt{2}x_3 \le 0\} \\
&\cup \{x_1 \le 0, x_1 + x_2 - \sqrt{2}x_3 \le 0\} \cup \{x_2 \le 0, x_1 + x_2 + \sqrt{2}x_3 \le 0\}
\end{aligned}$$

in $\mathbb{R}^3$ *for the standard* $\mathbb{R}^3$ *valued Brownian motion starting at the point* $(-2^{-1/2}, -2^{-1/2}, 0)$. *Hence there is no half space, which is the critical right cone for the power* $1/2$*, inscribed inside* $G_{2,2,2}$.

5. Two-sided capture time

In previous sections, we dealt with various capture times of prisoners by policemen all chasing from one side. In this section, we consider two policemen coming from both sides of a prisoner and prove Theorem 2.1. Indeed, we can compute the probability of the capture by given policeman and compare with the well-known first exit time σ_{ab} of $[a, b]$ for a Brownian motion. The fact that this capture time has finite expectation and its comparison with σ_{ab} were key ingredient in the invariance principle of Bramson and Griffeath (1980) for systems of coalescing or annihiliating random valks, and in Arratia's subsequent construction of stochastic flows of coalescing Brownian motions starting from every point of real line (see Arratia (1982) for details).

Proof of Theorem 1.2. Recall the joint representation

$$B_1(t) - B_0(t) = \sqrt{2}W_1(t), \quad B_2(t) - B_0(t) = 2^{-1/2}W_1(t) + (3/2)^{1/2}W_2(t)$$

given at the beginning of section 3. We have

$$\begin{aligned}
&\mathbb{P}\left(\tau_{ab} > t\right) \\
&\quad = \mathbb{P}\left(B_1(s) - B_0(s) < -a \text{ and } B_0(s) - B_2(s) < b \text{ for } 0 \le s \le t\right) \\
&\quad = \mathbb{P}\left(\sqrt{2}W_1(s) < -a,\ 2^{-1/2}W_1(s) + (3/2)^{1/2}W_2(s) > -b \text{ for } 0 \le s \le t\right) \\
&\quad = \Big(W_1(s) + a/\sqrt{2} < 0,\ (W_1(s) + a/\sqrt{2}) + 3^{1/2}(W_2(s) \\
&\qquad\quad + (2/3)^{1/2}(b - a/2)) > 0 \text{ for } 0 \le s \le t\Big) \\
&\quad = \mathbb{P}\left(\tau_G > t\right)
\end{aligned}$$

where τ_G is the first exit time of the domain

$$G = \{x = (x_1, x_2) \in \mathbb{R}^2 :\ x_1 \le 0,\ x_1 + \sqrt{3}x_2 \ge 0\} \tag{5.1}$$

for the standard $\mathbb{R}^2$ valued Brownian motion starting at the point $q = (2^{-1/2}a, (2/3)^{1/2}(b - a/2)) \in \mathbb{R}^2$.

The domain G in (5.1) is a cone with $2\theta = \pi/3$ and hence $p(\theta, n)/2 = 3/2$ for $n = 2$ and $\theta = \pi/6$. Our results (1.4) follow from the work of Spitzer (1958) as mentioned in section 2. In fact, $\tau_{ab} = -ab$ is given by (2.8) after an appropriate rotation of the region and the starting point to fit the formula. Furthermore, from the well-known formula for the probability that Brownian motion leaves an angular region through one of its two rays (see a derivation using only symmetry and continuity in Dynkin and Yushkevich (1969)) we have

$$\begin{aligned}\mathbb{P}\left(B_0(\tau_{ab}) - B_2(\tau_{ab}) = b\right) &= \frac{3}{\pi}\arctan\frac{-2^{-1/2}a}{(2/3)^{1/2}(b-a/2)}\\ &= \frac{3}{\pi}\arctan\frac{-\sqrt{3}a}{2b-a}\\ &= \frac{1}{2} - \frac{3}{\pi}\arctan\frac{a+b}{\sqrt{3}(b-a)}\end{aligned}$$

which finishes the proof.

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